Numbers and Sequences

Numbers

DEFINITION: **Rational numbers** are all numbers of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

EXAMPLE: $\frac{1}{2},\ -\frac{5}{3},\ 2,\ 0, \frac{50}{10}$, etc.

NOTATIONS:

$\mathbb{N}$ = all natural numbers, that is, $1, 2, 3, \ldots$

$\mathbb{Z}$ = all integer numbers, that is, $0, \pm 1, \pm 2, \pm 3, \ldots$

$\mathbb{Q}$ = all rational numbers

$\mathbb{R}$ = all real numbers

THE WELL-ORDERING PROPERTY: Every nonempty set of positive integers has a least element.

DEFINITION: A number which is not rational is said to be **irrational**.

EXAMPLE: Use the Well-Ordering Property to prove that $\sqrt{2}$ is irrational.

Proof: Assume to the contrary that $\sqrt{2}$ is rational, that is

$$\sqrt{2} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{2} \tag{3}$$

We now consider the following set:

$$S = \{k \mid k \text{ and } k\sqrt{2} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{2}$ are positive integers by (2) and (3), so b is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say, s. Clearly,

$$s \text{ and } s\sqrt{2} \text{ are positive integers} \tag{4}$$

since s is from S.

Consider the following number

$$w = s\sqrt{2} - s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) w is an integer, since s and $s\sqrt{2}$ are integers by (4) and w is the difference of these two integers. Moreover, w is a positive integer, since

$$w = s\sqrt{2} - s = s(\sqrt{2} - 1)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{2} - 1 > 0$.

(ii) $w\sqrt{2}$ is positive, because w is positive by (i). Moreover, $w\sqrt{2}$ is an integer, because

$$w\sqrt{2} = (s\sqrt{2} - s)\sqrt{2} = 2s - s\sqrt{2}$$

which is an integer, since s and $s\sqrt{2}$ are integers by (4).

(iii) $w < s$, since

$$w = s\sqrt{2} - s = s(\sqrt{2} - 1) < s$$

So, it immediately follows from (i) and (ii) that w is an element of S. But w is smaller than s by (iii). This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{2}$ is irrational. ■

REMARK 1: Here is a short version of the proof above: Assuming $\sqrt{2}$ is rational, let s be the smallest positive integer whose product with $\sqrt{2}$ is an integer. Then $s(\sqrt{2} - 1)$ is a smaller positive integer whose product with $\sqrt{2}$ is an integer, contradiction.

REMARK 2: Another version of the proof above can be found in Appendix I.

REMARK 3: To show that $\sqrt{3}$ is an irrational number, we just replace $\sqrt{2}$ by $\sqrt{3}$ everywhere in the proof above (see Appendix II, Example 1).

REMARK 4: It is *not* enough to replace $\sqrt{2}$ by $\sqrt{5}$ in the proof above to show that $\sqrt{5}$ is an irrational number. In addition, we should set

$$w = s\sqrt{5} - 2s$$

instead of setting $w = s\sqrt{5} - s$, since (iii) will fail otherwise (see Appendix II, Example 2).

REMARK 5: In order to show that, say, $\sqrt[3]{2}$ is irrational, we should consider the set

$$S = \{k \mid k,\ k\sqrt[3]{2} \text{ and } k\sqrt[3]{4} \text{ are positive integers}\}$$

and make other changes (see Appendix II, Example 3).

REMARK 6: It is known that π is an irrational number. However, we can't prove it in the same way as above. For example, in (i) we show that $s\sqrt{2}$ is a positive integer. Unfortunately, if we replace $\sqrt{2}$ by π, we'll get

$$s\pi = t\pi \cdot \pi = \pi^2 t$$

which is not an integer, since π^2 is not an integer.

Sequences

Stated formally, an **infinite sequence,** or more simply a **sequence,** is an unending succession of numbers, called **terms**.

EXAMPLES:

(a) $0,\ 0,\ 0,\ 0, \ldots$

(b) $1,\ 2,\ 3,\ 4,\ 5, \ldots$

(c) $2,\ 4,\ 6,\ 8,\ 10, \ldots$

(d) $1,\ 3,\ 5,\ 7,\ 9, \ldots$

(e) $1,\ -3,\ 5,\ -7,\ 9, \ldots$

(f) $0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1, \ldots$

(g) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}, \ldots$

(h) $1,\ 10,\ -\pi,\ \sqrt{3},\ 9.\overline{9},\ 100.7,\ -2/3, \ldots$

(i) $a_1,\ a_2,\ a_3,\ a_4,\ a_5, \ldots$

(j) $b_{-2},\ b_{-1},\ b_0,\ b_1,\ b_2, \ldots$

FAMOUS SEQUENCES:

1. Prime numbers: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots$

2. Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

3. Catalan numbers: $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$

DEFINITION: A **sequence** is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{\infty}$ to be an alternative notation for the function $f(n) = a_n,\ n = 1, 2, 3, \ldots$

EXAMPLE:

$$a_n = n \quad \text{or} \quad \{n\}_{n=1}^{\infty} \quad \text{or} \quad \{n\}$$

means $1,\ 2,\ 3,\ 4,\ 5, \ldots$

EXAMPLE:

$$a_n = \frac{1}{2^{n-1}} \quad \text{or} \quad \left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty} \quad \text{or} \quad \left\{\frac{1}{2^{n-1}}\right\}$$

means $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16}, \ldots$

EXAMPLE:

$$a_n = (-1)^{n+1}\frac{n}{2n+1} \quad \text{or} \quad \left\{(-1)^{n+1}\frac{n}{2n+1}\right\}_{n=1}^{\infty} \quad \text{or} \quad \left\{(-1)^{n+1}\frac{n}{2n+1}\right\}$$

means $\frac{1}{3},\ -\frac{2}{5},\ \frac{3}{7},\ -\frac{4}{9}, \ldots$

Finding Terms of Sequences

EXAMPLE: Write the first five terms of the sequence

$$a_n = 5 \text{ for all integers } n \geq 1$$

Solution: We have

$$a_1 = 5,\ a_2 = 5,\ a_3 = 5,\ a_4 = 5,\ a_5 = 5$$

EXAMPLE: Write the first five terms of the sequence

$$a_n = n \text{ for all integers } n \geq 1$$

Solution: We have

$$a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 4,\ a_5 = 5$$

EXAMPLE: Write the first five terms of the sequence

$$a_n = \frac{1}{n} \text{ for all integers } n \geq 4$$

Solution: We have

$$a_4 = \frac{1}{4},\ a_5 = \frac{1}{5},\ a_6 = \frac{1}{6},\ a_7 = \frac{1}{7},\ a_8 = \frac{1}{8}$$

EXAMPLE: Write the first five terms of the sequence

$$a_n = \sqrt{n+5} \text{ for all integers } n \geq -2$$

Solution: We have

$$a_{-2} = \sqrt{3},\ a_{-1} = \sqrt{4},\ a_0 = \sqrt{5},\ a_1 = \sqrt{6},\ a_2 = \sqrt{7}$$

EXAMPLE: Write the first five terms of the sequence

$$a_n = \frac{7+n^2}{10-n} \text{ for all integers } n \geq 1$$

Solution: We have

$$a_1 = \frac{7+1^2}{10-1} = \frac{8}{9},\ a_2 = \frac{7+2^2}{10-2} = \frac{11}{8},\ a_3 = \frac{7+3^2}{10-3} = \frac{16}{7},\ a_4 = \frac{7+4^2}{10-4} = \frac{23}{6},\ a_5 = \frac{7+5^2}{10-5} = \frac{32}{5}$$

EXAMPLE: Write the first five terms of the sequences

$$a_n = (-1)^n \frac{\sqrt{n+3}}{5+n} \text{ for all integers } n \geq 1$$

and

$$a_n = (-1)^{n+1} \frac{\sqrt{n+3}}{5+n} \text{ for all integers } n \geq 1$$

EXAMPLE: Write the first five terms of the sequences

$$a_n = (-1)^n \frac{\sqrt{n+3}}{5+n} \text{ for all integers } n \geq 1$$

and

$$a_n = (-1)^{n+1} \frac{\sqrt{n+3}}{5+n} \text{ for all integers } n \geq 1$$

Solution: We have

$$a_1 = (-1)^1 \frac{\sqrt{1+3}}{5+1} = -\frac{\sqrt{4}}{6}, \ a_2 = (-1)^2 \frac{\sqrt{2+3}}{5+2} = \frac{\sqrt{5}}{7}, \ a_3 = (-1)^3 \frac{\sqrt{3+3}}{5+3} = -\frac{\sqrt{6}}{8}$$

$$a_4 = (-1)^4 \frac{\sqrt{4+3}}{5+4} = \frac{\sqrt{7}}{9}, \ a_5 = (-1)^5 \frac{\sqrt{5+3}}{5+5} = -\frac{\sqrt{8}}{10}$$

and

$$a_1 = (-1)^2 \frac{\sqrt{1+3}}{5+1} = \frac{\sqrt{4}}{6}, \ a_2 = (-1)^3 \frac{\sqrt{2+3}}{5+2} = -\frac{\sqrt{5}}{7}, \ a_3 = (-1)^4 \frac{\sqrt{3+3}}{5+3} = \frac{\sqrt{6}}{8}$$

$$a_4 = (-1)^5 \frac{\sqrt{4+3}}{5+4} = -\frac{\sqrt{7}}{9}, \ a_5 = (-1)^6 \frac{\sqrt{5+3}}{5+5} = \frac{\sqrt{8}}{10}$$

Finding Explicit Formulas for Sequences

EXAMPLE: Find an explicit formula for the sequence of the form $a_1, \ a_2, \ a_3, \ldots$ with the initial terms

$$1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1$$

Solution: We have

$$a_n = 1 \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1, \ a_2, \ a_3, \ldots$ with the initial terms

$$1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, \ 8$$

Solution: We have

$$a_n = n \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1, \ a_2, \ a_3, \ldots$ with the initial terms

$$3, \ 4, \ 5, \ 6, \ 7, \ 8, \ 9, \ 10$$

Solution: We have

$$a_n = n + 2 \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$\frac{1}{7},\ \frac{1}{8},\ \frac{1}{9},\ \frac{1}{10},\ \frac{1}{11}$$

Solution: We have

$$a_n=\frac{1}{n+6}\text{ for all integers } n\ge 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$\frac{1}{12},\ \frac{1}{13},\ \frac{1}{14},\ \frac{1}{15},\ \frac{1}{16}$$

Solution: We have

$$a_n=\frac{1}{n+11}\text{ for all integers } n\ge 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$\sqrt[3]{4},\ \sqrt[3]{9},\ \sqrt[3]{16},\ \sqrt[3]{25}$$

Solution: We have

$$a_n=\sqrt[3]{(n+1)^2}\text{ for all integers } n\ge 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$-\sqrt[3]{4},\ \sqrt[3]{9},\ -\sqrt[3]{16},\ \sqrt[3]{25}$$

Solution: We have

$$a_n=(-1)^n\sqrt[3]{(n+1)^2}\text{ for all integers } n\ge 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$\sqrt[3]{4},\ -\sqrt[3]{9},\ \sqrt[3]{16},\ -\sqrt[3]{25}$$

Solution: We have

$$a_n=(-1)^{n+1}\sqrt[3]{(n+1)^2}\text{ for all integers } n\ge 1$$

EXAMPLE: Find explicit formulas for the sequences of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

(a) $-\frac{3}{8},\ \frac{4}{10},\ -\frac{5}{12},\ \frac{6}{14},\ -\frac{7}{16}$ and $\frac{3}{8},\ -\frac{4}{10},\ \frac{5}{12},\ -\frac{6}{14},\ \frac{7}{16}$

(b) $1,\ 2,\ 6,\ 24,\ 120$

(c) $4,\ 12,\ 28,\ 60,\ 124$

(d) $\sqrt[3]{7},\ \sqrt[3]{11},\ \sqrt[3]{15},\ \sqrt[3]{19},\ \sqrt[3]{23},\ \sqrt[3]{27}$

(e) $\sin 5,\ \sin 10,\ \sin 17,\ \sin 26,\ \sin 37$

(f) $-4,\ 3,\ 22,\ 59,\ 120$

(g) $0,\ 1,\ 0,\ 1,\ 0,\ 1$

EXAMPLE: Find explicit formulas for the sequences of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

(a) $-\dfrac{3}{8},\ \dfrac{4}{10},\ -\dfrac{5}{12},\ \dfrac{6}{14},\ -\dfrac{7}{16}$ and $\dfrac{3}{8},\ -\dfrac{4}{10},\ \dfrac{5}{12},\ -\dfrac{6}{14},\ \dfrac{7}{16}$

Answer: $a_n = (-1)^n \dfrac{n+2}{2(n+3)}$ and $a_n = (-1)^{n+1}\dfrac{n+2}{2(n+3)}$, respectively, for all integers $n \geq 1$.

(b) $1,\ 2,\ 6,\ 24,\ 120$ **Answer:** $a_n = n!$ for all integers $n \geq 1$.

(c) $4,\ 12,\ 28,\ 60,\ 124$ **Answer:** $a_n = 2^{n+2} - 4 = 4(2^n - 1)$ for all integers $n \geq 1$.

(d) $\sqrt[3]{7},\ \sqrt[3]{11},\ \sqrt[3]{15},\ \sqrt[3]{19},\ \sqrt[3]{23},\ \sqrt[3]{27}$ **Answer:** $a_n = \sqrt[3]{4n+3}$ for all integers $n \geq 1$.

(e) $\sin 5,\ \sin 10,\ \sin 17,\ \sin 26,\ \sin 37$ **Answer:** $a_n = \sin((n+1)^2 + 1)$ for all integers $n \geq 1$.

(f) $-4,\ 3,\ 22,\ 59,\ 120$ **Answer:** $a_n = n^3 - 5$ for all integers $n \geq 1$.

(g) $0,\ 1,\ 0,\ 1,\ 0,\ 1$ **Answer:** $a_n = \dfrac{(-1)^n + 1}{2}$ or $\left|\sin\left((n-1)\dfrac{\pi}{2}\right)\right|$ for all integers $n \geq 1$.

Appendix I

EXAMPLE: Use the Well-Ordering Property to prove that $\sqrt{2}$ is irrational.

Proof (version 2): Assume to the contrary that $\sqrt{2}$ is rational, that is

$$\sqrt{2} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{2} \tag{3}$$

We now consider the following set:

$$S = \{k\sqrt{2} \mid k \text{ and } k\sqrt{2} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{2}$ are positive integers by (2) and (3), so $b\sqrt{2}$ is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say,

$$s = t\sqrt{2}, \quad \text{where } s \text{ and } t \text{ are positive integers} \tag{4}$$

Consider the following number

$$w = s\sqrt{2} - s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) $s\sqrt{2}$ is a positive integer, since

$$s\sqrt{2} \overset{(4)}{=} t\sqrt{2} \cdot \sqrt{2} = 2t$$

which is a positive integer, since t is a positive integer by (4).

(ii) w is an integer, since s is an integer by (4), $s\sqrt{2}$ is an integer by (i) and w is the difference of these two integers. Moreover, it is a positive integer, since

$$w = s\sqrt{2} - s = s(\sqrt{2} - 1)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{2} - 1 > 0$.

(iii) w is a member of S. To show that, we first observe that

$$w = s\sqrt{2} - s \overset{(4)}{=} s\sqrt{2} - t\sqrt{2} = (s - t)\sqrt{2}$$

therefore w is of the form $k\sqrt{2}$ with $k = s - t$. Note that both k and $k\sqrt{2}$ are positive integers. Indeed, k is an integer, because s and t are integers. Moreover, k is positive, because

$$k = s - t = \frac{w}{\sqrt{2}} \overset{\text{(ii)}}{>} 0$$

Finally, $k\sqrt{2}$ is a positive integer, since $k\sqrt{2} = w$ and w is a positive integer by (ii).

(iv) $w < s$, since

$$w = s\sqrt{2} - s = s(\sqrt{2} - 1) < s$$

So, it immediately follows from (iii) and (iv) that w is an element of S which is smaller than s. This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{2}$ is irrational. ■

Appendix II

EXAMPLE 1: Use the Well-Ordering Property to prove that $\sqrt{3}$ is irrational.

Proof (version 1): Assume to the contrary that $\sqrt{3}$ is rational, that is

$$\sqrt{3} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{3} \tag{3}$$

We now consider the following set:

$$S = \{k \mid k \text{ and } k\sqrt{3} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{3}$ are positive integers by (2) and (3), so b is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say, s. Clearly,

$$s \text{ and } s\sqrt{3} \text{ are positive integers} \tag{4}$$

since s is from S.

Consider the following number

$$w = s\sqrt{3} - s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) w is an integer, since s and $s\sqrt{3}$ are integers by (4) and w is the difference of these two integers. Moreover, w is a positive integer, since

$$w = s\sqrt{3} - s = s(\sqrt{3} - 1)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{3} - 1 > 0$.

(ii) $w\sqrt{3}$ is positive, because w is positive by (i). Moreover, $w\sqrt{3}$ is an integer, because

$$w\sqrt{3} = (s\sqrt{3} - s)\sqrt{3} = 3s - s\sqrt{3}$$

which is an integer, since s and $s\sqrt{3}$ are integers by (4).

(iii) $w < s$, since

$$w = s\sqrt{3} - s = s(\sqrt{3} - 1) < s$$

So, it immediately follows from (i) and (ii) that w is an element of S. But w is smaller than s by (iii). This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{3}$ is irrational. ■

Proof (version 2): Assume to the contrary that $\sqrt{3}$ is rational, that is

$$\sqrt{3} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{3} \tag{3}$$

We now consider the following set:

$$S = \{k\sqrt{3} \mid k \text{ and } k\sqrt{3} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{3}$ are positive integers by (2) and (3), so $b\sqrt{3}$ is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say,

$$s = t\sqrt{3}, \qquad \text{where } s \text{ and } t \text{ are positive integers} \tag{4}$$

Consider the following number

$$w = s\sqrt{3} - s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) $s\sqrt{3}$ is a positive integer, since

$$s\sqrt{3} \overset{(4)}{=} t\sqrt{3} \cdot \sqrt{3} = 3t$$

which is a positive integer, since t is a positive integer by (4).

(ii) w is an integer, since s is an integer by (4), $s\sqrt{3}$ is an integer by (i) and w is the difference of these two integers. Moreover, it is a positive integer, since

$$w = s\sqrt{3} - s = s(\sqrt{3} - 1)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{3} - 1 > 0$.

(iii) w is a member of S. To show that, we first observe that

$$w = s\sqrt{3} - s \overset{(4)}{=} s\sqrt{3} - t\sqrt{3} = (s - t)\sqrt{3}$$

therefore w is of the form $k\sqrt{3}$ with $k = s - t$. Note that both k and $k\sqrt{3}$ are positive integers. Indeed, k is an integer, because s and t are integers. Moreover, k is positive, because

$$k = s - t = \frac{w}{\sqrt{3}} \overset{\text{(ii)}}{>} 0$$

Finally, $k\sqrt{3}$ is a positive integer, since $k\sqrt{3} = w$ and w is a positive integer by (ii).

(iv) $w < s$, since

$$w = s\sqrt{3} - s = s(\sqrt{3} - 1) < s$$

So, it immediately follows from (iii) and (iv) that w is an element of S which is smaller than s. This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{3}$ is irrational. ■

EXAMPLE 2: Use the Well-Ordering Property to prove that $\sqrt{5}$ is irrational.

Proof (version 1): Assume to the contrary that $\sqrt{5}$ is rational, that is

$$\sqrt{5} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{5} \tag{3}$$

We now consider the following set:

$$S = \{k \mid k \text{ and } k\sqrt{5} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{5}$ are positive integers by (2) and (3), so b is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say, s. Clearly,

$$s \text{ and } s\sqrt{5} \text{ are positive integers} \tag{4}$$

since s is from S.

Consider the following number

$$w = s\sqrt{5} - 2s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) w is an integer, since $2s$ and $s\sqrt{5}$ are integers by (4) and w is the difference of these two integers. Moreover, w is a positive integer, since

$$w = s\sqrt{5} - 2s = s(\sqrt{5} - 2)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{5} - 2 > 0$.

(ii) $w\sqrt{5}$ is positive, because w is positive by (i). Moreover, $w\sqrt{5}$ is an integer, because

$$w\sqrt{5} = (s\sqrt{5} - 2s)\sqrt{5} = 5s - 2s\sqrt{5}$$

which is an integer, since s and $s\sqrt{5}$ are integers by (4).

(iii) $w < s$, since

$$w = s\sqrt{5} - 2s = s(\sqrt{5} - 2) < s$$

So, it immediately follows from (i) and (ii) that w is an element of S. But w is smaller than s by (iii). This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{5}$ is irrational. ■

Proof (version 2): Assume to the contrary that $\sqrt{5}$ is rational, that is

$$\sqrt{5} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{5} \tag{3}$$

We now consider the following set:

$$S = \{k\sqrt{5} \mid k \text{ and } k\sqrt{5} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{5}$ are positive integers by (2) and (3), so $b\sqrt{5}$ is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say,

$$s = t\sqrt{5}, \qquad \text{where } s \text{ and } t \text{ are positive integers} \tag{4}$$

Consider the following number

$$w = s\sqrt{5} - 2s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) $s\sqrt{5}$ is a positive integer, since

$$s\sqrt{5} \overset{(4)}{=} t\sqrt{5} \cdot \sqrt{5} = 5t$$

which is a positive integer, since t is a positive integer by (4).

(ii) w is an integer, since $2s$ is an integer by (4), $s\sqrt{5}$ is an integer by (i) and w is the difference of these two integers. Moreover, it is a positive integer, since

$$w = s\sqrt{5} - 2s = s(\sqrt{5} - 2)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{5} - 2 > 0$.

(iii) w is a member of S. To show that, we first observe that

$$w = s\sqrt{5} - 2s \overset{(4)}{=} s\sqrt{5} - 2t\sqrt{5} = (s - 2t)\sqrt{5}$$

therefore w is of the form $k\sqrt{5}$ with $k = s - 2t$. Note that both k and $k\sqrt{5}$ are positive integers. Indeed, k is an integer, because s and t are integers. Moreover, k is positive, because

$$k = s - 2t = \frac{w}{\sqrt{5}} \overset{\text{(ii)}}{>} 0$$

Finally, $k\sqrt{5}$ is a positive integer, since $k\sqrt{5} = w$ and w is a positive integer by (ii).

(iv) $w < s$, since

$$w = s\sqrt{5} - 2s = s(\sqrt{5} - 2) < s$$

So, it immediately follows from (iii) and (iv) that w is an element of S which is smaller than s. This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{5}$ is irrational. ∎

EXAMPLE 3: Use the Well-Ordering Property to prove that $\sqrt[3]{2}$ is irrational.

Proof (version 1): Assume to the contrary that $\sqrt[3]{2}$ is rational, that is

$$\sqrt[3]{2} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt[3]{2} \tag{3}$$

We now consider the following set:

$$S = \{k \mid k,\ k\sqrt[3]{2} \text{ and } k\sqrt[3]{4} \text{ are positive integers}\}$$

Note that S is nonempty. Indeed, since b and $b\sqrt[3]{2}$ are positive integers by (2) and (3), it follows that b^2, $b^2\sqrt[3]{2}$ are positive integers. But $a^2 = b^2\sqrt[3]{4}$ by (3), therefore $b^2\sqrt[3]{4}$ is also a positive integer. So, b^2, $b^2\sqrt[3]{2}$ and $b^2\sqrt[3]{4}$ are positive integers, therefore b^2 is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say, s. Clearly,

$$s,\ s\sqrt[3]{2} \text{ and } s\sqrt[3]{4} \text{ are positive integers} \tag{4}$$

since s is from S.

Consider the following number

$$w = s\sqrt[3]{2} - s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) w is a positive integer. Indeed, w is an integer, since s and $s\sqrt[3]{2}$ are integers by (4) and w is the difference of these two integers. Moreover, w is a positive integer, since

$$w = s\sqrt[3]{2} - s = s(\sqrt[3]{2} - 1)$$

which is > 0, because $s > 0$ by (4) and $\sqrt[3]{2} - 1 > 0$.

(ii) $w\sqrt[3]{2}$ is a positive integer. Indeed, $w\sqrt[3]{2}$ is positive, because w is positive by (i). Moreover, $w\sqrt[3]{2}$ is an integer, because

$$w\sqrt[3]{2} = (s\sqrt[3]{2} - s)\sqrt[3]{2} = s\sqrt[3]{2}\sqrt[3]{2} - s\sqrt[3]{2} = s\sqrt[3]{4} - s\sqrt[3]{2}$$

which is an integer, since $s\sqrt[3]{4}$ and $s\sqrt[3]{2}$ are integers by (4).

(iii) $w\sqrt[3]{4}$ is a positive integer. Indeed, $w\sqrt[3]{4}$ is positive, because w is positive by (i). Moreover, $w\sqrt[3]{4}$ is an integer, because

$$w\sqrt[3]{4} = (s\sqrt[3]{2} - s)\sqrt[3]{4} = s\sqrt[3]{2}\sqrt[3]{4} - s\sqrt[3]{4} = 2s - s\sqrt[3]{4}$$

which is an integer, since s and $s\sqrt[3]{4}$ are integers by (4).

(iv) $w < s$, since

$$w = s\sqrt[3]{2} - s = s(\sqrt[3]{2} - 1) < s$$

So, $w, w\sqrt[3]{2}, w\sqrt[3]{4}$ are positive integers by (i)-(iii). Therefore w is an element of S. But w is smaller than s by (iv). This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt[3]{2}$ is irrational. ■

Proof (version 2): Assume to the contrary that $\sqrt[3]{2}$ is rational, that is

$$\sqrt[3]{2} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt[3]{2} \tag{3}$$

We now consider the following set:

$$S = \{k\sqrt[3]{2} \mid k,\ k\sqrt[3]{2} \text{ and } k\sqrt[3]{4} \text{ are positive integers}\}$$

Note that S is nonempty. Indeed, since b and $b\sqrt[3]{2}$ are positive integers by (2) and (3), it follows that b^2, $b^2\sqrt[3]{2}$ are positive integers. But $a^2 = b^2\sqrt[3]{4}$ by (3), therefore $b^2\sqrt[3]{4}$ is also a positive integer. So, b^2, $b^2\sqrt[3]{2}$ and $b^2\sqrt[3]{4}$ are positive integers, therefore $b^2\sqrt[3]{2}$ is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say,

$$s = t\sqrt[3]{2}, \quad \text{where } s, t \text{ and } t\sqrt[3]{4} \text{ are positive integers} \tag{4}$$

Consider the following number

$$w = s\sqrt[3]{2} - s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) $s\sqrt[3]{2}$ is a positive integer, since

$$s\sqrt[3]{2} \overset{(4)}{=} t\sqrt[3]{2} \cdot \sqrt[3]{2} = t\sqrt[3]{4}$$

which is a positive integer, since $t\sqrt[3]{4}$ is a positive integer by (4).

(ii) w is an integer, since s is an integer by (4), $s\sqrt[3]{2}$ is an integer by (i) and w is the difference of these two integers. Moreover, it is a positive integer, since

$$w = s\sqrt[3]{2} - s = s(\sqrt[3]{2} - 1)$$

which is > 0, because $s > 0$ by (4) and $\sqrt[3]{2} - 1 > 0$.

(iii) w is a member of S. To show that, we first observe that

$$w = s\sqrt[3]{2} - s \overset{(4)}{=} s\sqrt[3]{2} - t\sqrt[3]{2} = (s-t)\sqrt[3]{2}$$

therefore w is of the form $k\sqrt[3]{2}$ with $k = s - t$. Note that k, $k\sqrt[3]{2}$ and $k\sqrt[3]{4}$ are positive integers. Indeed,

(a) k is an integer, because s and t are integers. Moreover, k is positive, because

$$k = s - t = \frac{w}{\sqrt[3]{2}} \overset{\text{(ii)}}{>} 0$$

(b) $k\sqrt[3]{2}$ is a positive integer, since $k\sqrt[3]{2} = w$ and w is a positive integer by (ii).

(c) $k\sqrt[3]{4}$ is positive, since k is positive by (a). Moreover, $k\sqrt[3]{4}$ is an integer, since

$$k\sqrt[3]{4} = (s-t)\sqrt[3]{4} \stackrel{(4)}{=} (t\sqrt[3]{2}-t)\sqrt[3]{4} = 2t - t\sqrt[3]{4}$$

which is an integer, because t and $t\sqrt[3]{4}$ are integers by (4).

(iv) $w < s$, since

$$w = s\sqrt[3]{2} - s = s(\sqrt[3]{2}-1) < s$$

So, it immediately follows from (iii) and (iv) that w is an element of S which is smaller than s. This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt[3]{2}$ is irrational. ■

EXAMPLE 4: Use the Well-Ordering Property to prove that $\sqrt{15}$ is irrational.

Proof (version 1): Assume to the contrary that $\sqrt{15}$ is rational, that is

$$\sqrt{15} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{15} \tag{3}$$

We now consider the following set:

$$S = \{k \mid k \text{ and } k\sqrt{15} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{15}$ are positive integers by (2) and (3), so b is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say, s. Clearly,

$$s \text{ and } s\sqrt{15} \text{ are positive integers} \tag{4}$$

since s is from S.

Consider the following number

$$w = s\sqrt{15} - 3s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) w is an integer, since $3s$ and $s\sqrt{15}$ are integers by (4) and w is the difference of these two integers. Moreover, w is a positive integer, since

$$w = s\sqrt{15} - 3s = s(\sqrt{15}-3)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{15} - 3 > 0$.

(ii) $w\sqrt{15}$ is positive, because w is positive by (i). Moreover, $w\sqrt{15}$ is an integer, because

$$w\sqrt{15} = (s\sqrt{15} - 3s)\sqrt{15} = 15s - 3s\sqrt{15}$$

which is an integer, since s and $s\sqrt{15}$ are integers by (4).

(iii) $w < s$, since

$$w = s\sqrt{15} - 3s = s(\sqrt{15}-3) < s$$

So, it immediately follows from (i) and (ii) that w is an element of S. But w is smaller than s by (iii). This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{15}$ is irrational. ■

Proof (version 2): Assume to the contrary that $\sqrt{15}$ is rational, that is

$$\sqrt{15} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{15} \tag{3}$$

We now consider the following set:

$$S = \{k\sqrt{15} \mid k \text{ and } k\sqrt{15} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{15}$ are positive integers by (2) and (3), so $b\sqrt{15}$ is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say,

$$s = t\sqrt{15}, \qquad \text{where } s \text{ and } t \text{ are positive integers} \tag{4}$$

Consider the following number

$$w = s\sqrt{15} - 3s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) $s\sqrt{15}$ is a positive integer, since

$$s\sqrt{15} \overset{(4)}{=} t\sqrt{15} \cdot \sqrt{15} = 15t$$

which is a positive integer, since t is a positive integer by (4).

(ii) w is an integer, since $3s$ is an integer by (4), $s\sqrt{15}$ is an integer by (i) and w is the difference of these two integers. Moreover, it is a positive integer, since

$$w = s\sqrt{15} - 3s = s(\sqrt{15} - 3)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{15} - 3 > 0$.

(iii) w is a member of S. To show that, we first observe that

$$w = s\sqrt{15} - 3s \overset{(4)}{=} s\sqrt{15} - 3t\sqrt{15} = (s - 3t)\sqrt{15}$$

therefore w is of the form $k\sqrt{15}$ with $k = s - 3t$. Note that both k and $k\sqrt{15}$ are positive integers. Indeed, k is an integer, because s and t are integers. Moreover, k is positive, because

$$k = s - 3t = \frac{w}{\sqrt{15}} \overset{(ii)}{>} 0$$

Finally, $k\sqrt{15}$ is a positive integer, since $k\sqrt{15} = w$ and w is a positive integer by (ii).

(iv) $w < s$, since

$$w = s\sqrt{15} - 3s = s(\sqrt{15} - 3) < s$$

So, it immediately follows from (iii) and (iv) that w is an element of S which is smaller than s. This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{15}$ is irrational. ■

REMARK: This problem was given as a Midterm Exam question in Summer of 2017.