Numbers and Sequences

Numbers

DEFINITION: **Rational numbers** are all numbers of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

EXAMPLE: $\frac{1}{2},\ -\frac{5}{3},\ 2,\ 0, \frac{50}{10}$, etc.

NOTATIONS:

$\mathbb{N}$ = all natural numbers, that is, $1, 2, 3, \ldots$

$\mathbb{Z}$ = all integer numbers, that is, $0, \pm 1, \pm 2, \pm 3, \ldots$

$\mathbb{Q}$ = all rational numbers

$\mathbb{R}$ = all real numbers

THE WELL-ORDERING PROPERTY: Every nonempty set of positive integers has a least element.

DEFINITION: A number which is not rational is said to be **irrational**.

THEOREM: $\sqrt{2}$ is irrational.

Proof: Assume to the contrary that $\sqrt{2}$ is rational, that is

$$\sqrt{2} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{2} \tag{3}$$

We now consider the following set:

$$S = \{k\sqrt{2} \mid k \text{ and } k\sqrt{2} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{2}$ are positive integers by (2) and (3) and therefore $b\sqrt{2}$ is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say,

$$s = t\sqrt{2}, \quad \text{where } s \text{ and } t \text{ are positive integers} \tag{4}$$

Consider the following number

$$w = s\sqrt{2} - s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) $s\sqrt{2}$ is a positive integer, since

$$s\sqrt{2} \overset{(4)}{=} t\sqrt{2} \cdot \sqrt{2} = 2t$$

which is a positive integer, since t is a positive integer by (4).

(ii) w is an integer, since s is an integer by (4), $s\sqrt{2}$ is an integer by (i) and w is the difference of these two integers. Moreover, it is a *positive* integer, since

$$w = s\sqrt{2} - s = s(\sqrt{2} - 1)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{2} - 1 > 0$.

(iii) w is a member of S. To show that, we first observe that

$$w = s\sqrt{2} - s \overset{(4)}{=} s\sqrt{2} - t\sqrt{2} = (s - t)\sqrt{2}$$

therefore w is of the form $k\sqrt{2}$ with $k = s - t$. Moreover, $k\sqrt{2}$ is a positive integer, since w is a positive integer by (ii) and $k\sqrt{2} = w$.

(iv) $w < s$, since

$$w = s\sqrt{2} - s = s(\sqrt{2} - 1) < s$$

So, it immediately follows from (iii) and (iv) that w is an element of S which is smaller than s. This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{2}$ is irrational.

REMARK 1: To show that $\sqrt{3}$ is an irrational number, we just replace $\sqrt{2}$ by $\sqrt{3}$ everywhere in the above proof.

REMARK 2: It is *not* enough to replace $\sqrt{2}$ by $\sqrt{5}$ in the above proof to show that $\sqrt{5}$ is an irrational number. In addition, we should set

$$w = s\sqrt{5} - 2s$$

instead of setting $w = s\sqrt{5} - s$, since (iv) will fail otherwise.

Similarly, to show that, say, $\sqrt[3]{2}$ is irrational, we should replace $\sqrt{2}$ by $\sqrt[3]{2}$ everywhere in the above prove *and* set

$$s = t\sqrt[3]{4}$$

in (4) instead of setting $s = t\sqrt[3]{2}$, since (i) will fail otherwise. For more, see the Appendix.

REMARK 3: It is known that π is an irrational number. However, we can't prove it in the same way as above. Indeed, in (i) we show that $s\sqrt{2}$ is a positive integer. Unfortunately, if we replace $\sqrt{2}$ by π, we'll get

$$s\pi \overset{(4)}{=} t\pi \cdot \pi = \pi^2 t$$

which is not an integer, since π^2 is not an integer.

Sequences

Stated formally, an **infinite sequence,** or more simply a **sequence,** is an unending succession of numbers, called **terms**.

EXAMPLES:

(a) $0,\ 0,\ 0,\ 0, \dots$

(b) $1,\ 2,\ 3,\ 4,\ 5, \dots$

(c) $2,\ 4,\ 6,\ 8,\ 10, \dots$

(d) $1,\ 3,\ 5,\ 7,\ 9, \dots$

(e) $1,\ -3,\ 5,\ -7,\ 9, \dots$

(f) $0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1, \dots$

(g) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}, \dots$

(h) $1,\ 10,\ -\pi,\ \sqrt{3},\ 9.\overline{9},\ 100.7,\ -2/3, \dots$

(i) $a_1,\ a_2,\ a_3,\ a_4,\ a_5, \dots$

(j) $b_{-2},\ b_{-1},\ b_0,\ b_1,\ b_2, \dots$

FAMOUS SEQUENCES:

1. Prime numbers: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \dots$

2. Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$

3. Catalan numbers: $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \dots$

DEFINITION: A **sequence** is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{\infty}$ to be an alternative notation for the function $f(n) = a_n,\ n = 1, 2, 3, \dots$

EXAMPLE:

$$a_n = n \quad \text{or} \quad \{n\}_{n=1}^{\infty} \quad \text{or} \quad \{n\}$$

means $1,\ 2,\ 3,\ 4,\ 5, \dots$

EXAMPLE:

$$a_n = \frac{1}{2^{n-1}} \quad \text{or} \quad \left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty} \quad \text{or} \quad \left\{\frac{1}{2^{n-1}}\right\}$$

means $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16}, \dots$

EXAMPLE:

$$a_n = (-1)^{n+1}\frac{n}{2n+1} \quad \text{or} \quad \left\{(-1)^{n+1}\frac{n}{2n+1}\right\}_{n=1}^{\infty} \quad \text{or} \quad \left\{(-1)^{n+1}\frac{n}{2n+1}\right\}$$

means $\frac{1}{3},\ -\frac{2}{5},\ \frac{3}{7},\ -\frac{4}{9}, \dots$

Finding Terms of Sequences

EXAMPLE: Write the first five terms of the sequence

$$a_n = 5 \text{ for all integers } n \geq 1$$

Solution: We have

$$a_1 = 5,\ a_2 = 5,\ a_3 = 5,\ a_4 = 5,\ a_5 = 5$$

EXAMPLE: Write the first five terms of the sequence

$$a_n = n \text{ for all integers } n \geq 1$$

Solution: We have

$$a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 4,\ a_5 = 5$$

EXAMPLE: Write the first five terms of the sequence

$$a_n = \frac{1}{n} \text{ for all integers } n \geq 4$$

Solution: We have

$$a_4 = \frac{1}{4},\ a_5 = \frac{1}{5},\ a_6 = \frac{1}{6},\ a_7 = \frac{1}{7},\ a_8 = \frac{1}{8}$$

EXAMPLE: Write the first five terms of the sequence

$$a_n = \sqrt{n+5} \text{ for all integers } n \geq -2$$

Solution: We have

$$a_{-2} = \sqrt{3},\ a_{-1} = \sqrt{4},\ a_0 = \sqrt{5},\ a_1 = \sqrt{6},\ a_2 = \sqrt{7}$$

EXAMPLE: Write the first five terms of the sequence

$$a_n = \frac{7+n^2}{10-n} \text{ for all integers } n \geq 1$$

Solution: We have

$$a_1 = \frac{7+1^2}{10-1} = \frac{8}{9},\ a_2 = \frac{7+2^2}{10-2} = \frac{11}{8},\ a_3 = \frac{7+3^2}{10-3} = \frac{16}{7},\ a_4 = \frac{7+4^2}{10-4} = \frac{23}{6},\ a_5 = \frac{7+5^2}{10-5} = \frac{32}{5}$$

EXAMPLE: Write the first five terms of the sequences

$$a_n = (-1)^n \frac{\sqrt{n+3}}{5+n} \text{ for all integers } n \geq 1$$

and

$$a_n = (-1)^{n+1} \frac{\sqrt{n+3}}{5+n} \text{ for all integers } n \geq 1$$

EXAMPLE: Write the first five terms of the sequences

$$a_n = (-1)^n \frac{\sqrt{n+3}}{5+n} \text{ for all integers } n \geq 1$$

and

$$a_n = (-1)^{n+1} \frac{\sqrt{n+3}}{5+n} \text{ for all integers } n \geq 1$$

Solution: We have

$$a_1 = (-1)^1 \frac{\sqrt{1+3}}{5+1} = -\frac{\sqrt{4}}{6}, \ a_2 = (-1)^2 \frac{\sqrt{2+3}}{5+2} = \frac{\sqrt{5}}{7}, \ a_3 = (-1)^3 \frac{\sqrt{3+3}}{5+3} = -\frac{\sqrt{6}}{8}$$

$$a_4 = (-1)^4 \frac{\sqrt{4+3}}{5+4} = \frac{\sqrt{7}}{9}, \ a_5 = (-1)^5 \frac{\sqrt{5+3}}{5+5} = -\frac{\sqrt{8}}{10}$$

and

$$a_1 = (-1)^2 \frac{\sqrt{1+3}}{5+1} = \frac{\sqrt{4}}{6}, \ a_2 = (-1)^3 \frac{\sqrt{2+3}}{5+2} = -\frac{\sqrt{5}}{7}, \ a_3 = (-1)^4 \frac{\sqrt{3+3}}{5+3} = \frac{\sqrt{6}}{8}$$

$$a_4 = (-1)^5 \frac{\sqrt{4+3}}{5+4} = -\frac{\sqrt{7}}{9}, \ a_5 = (-1)^6 \frac{\sqrt{5+3}}{5+5} = \frac{\sqrt{8}}{10}$$

Finding Explicit Formulas for Sequences

EXAMPLE: Find an explicit formula for the sequence of the form $a_1, \ a_2, \ a_3, \ldots$ with the initial terms

$$1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1$$

Solution: We have

$$a_n = 1 \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1, \ a_2, \ a_3, \ldots$ with the initial terms

$$1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, \ 8$$

Solution: We have

$$a_n = n \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1, \ a_2, \ a_3, \ldots$ with the initial terms

$$3, \ 4, \ 5, \ 6, \ 7, \ 8, \ 9, \ 10$$

Solution: We have

$$a_n = n + 2 \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$\frac{1}{7},\ \frac{1}{8},\ \frac{1}{9},\ \frac{1}{10},\ \frac{1}{11}$$

Solution: We have

$$a_n = \frac{1}{n+6} \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$\frac{1}{12},\ \frac{1}{13},\ \frac{1}{14},\ \frac{1}{15},\ \frac{1}{16}$$

Solution: We have

$$a_n = \frac{1}{n+11} \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$\sqrt[3]{4},\ \sqrt[3]{9},\ \sqrt[3]{16},\ \sqrt[3]{25}$$

Solution: We have

$$a_n = \sqrt[3]{(n+1)^2} \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$-\sqrt[3]{4},\ \sqrt[3]{9},\ -\sqrt[3]{16},\ \sqrt[3]{25}$$

Solution: We have

$$a_n = (-1)^n\sqrt[3]{(n+1)^2} \text{ for all integers } n \geq 1$$

EXAMPLE: Find an explicit formula for the sequence of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

$$\sqrt[3]{4},\ -\sqrt[3]{9},\ \sqrt[3]{16},\ -\sqrt[3]{25}$$

Solution: We have

$$a_n = (-1)^{n+1}\sqrt[3]{(n+1)^2} \text{ for all integers } n \geq 1$$

EXAMPLE: Find explicit formulas for the sequences of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

(a) $-\frac{3}{8},\ \frac{4}{10},\ -\frac{5}{12},\ \frac{6}{14},\ -\frac{7}{16}$ and $\frac{3}{8},\ -\frac{4}{10},\ \frac{5}{12},\ -\frac{6}{14},\ \frac{7}{16}$

(b) $1,\ 2,\ 6,\ 24,\ 120$

(c) $4,\ 12,\ 28,\ 60,\ 124$

(d) $\sqrt[3]{7},\ \sqrt[3]{11},\ \sqrt[3]{15},\ \sqrt[3]{19},\ \sqrt[3]{23},\ \sqrt[3]{27}$

(e) $\sin 5,\ \sin 10,\ \sin 17,\ \sin 26,\ \sin 37$

(f) $-4,\ 3,\ 22,\ 59,\ 120$

(g) $0,\ 1,\ 0,\ 1,\ 0,\ 1$

EXAMPLE: Find explicit formulas for the sequences of the form $a_1,\ a_2,\ a_3,\ldots$ with the initial terms

(a) $-\dfrac{3}{8},\ \dfrac{4}{10},\ -\dfrac{5}{12},\ \dfrac{6}{14},\ -\dfrac{7}{16}$ and $\dfrac{3}{8},\ -\dfrac{4}{10},\ \dfrac{5}{12},\ -\dfrac{6}{14},\ \dfrac{7}{16}$

Answer: $a_n=(-1)^n\dfrac{n+2}{2(n+3)}$ and $a_n=(-1)^{n+1}\dfrac{n+2}{2(n+3)}$, respectively, for all integers $n\ge 1$.

(b) $1,\ 2,\ 6,\ 24,\ 120$ **Answer:** $a_n=n!$ for all integers $n\ge 1$.

(c) $4,\ 12,\ 28,\ 60,\ 124$ **Answer:** $a_n=2^{n+2}-4=4(2^n-1)$ for all integers $n\ge 1$.

(d) $\sqrt[3]{7},\ \sqrt[3]{11},\ \sqrt[3]{15},\ \sqrt[3]{19},\ \sqrt[3]{23},\ \sqrt[3]{27}$ **Answer:** $a_n=\sqrt[3]{4n+3}$ for all integers $n\ge 1$.

(e) $\sin 5,\ \sin 10,\ \sin 17,\ \sin 26,\ \sin 37$ **Answer:** $a_n=\sin((n+1)^2+1)$ for all integers $n\ge 1$.

(f) $-4,\ 3,\ 22,\ 59,\ 120$ **Answer:** $a_n=n^3-5$ for all integers $n\ge 1$.

(g) $0,\ 1,\ 0,\ 1,\ 0,\ 1$ **Answer:** $a_n=\dfrac{(-1)^n+1}{2}$ or $\left|\sin\left((n-1)\dfrac{\pi}{2}\right)\right|$ for all integers $n\ge 1$.

Appendix

EXAMPLE: $\sqrt{15}$ is irrational.

Proof: Assume to the contrary that $\sqrt{15}$ is rational, that is

$$\sqrt{15} = \frac{a}{b} \tag{1}$$

where a and b are integers and $b \neq 0$. Without loss of generality we can assume that

$$a \text{ and } b \text{ are positive integers} \tag{2}$$

It immediately follows from (1) that

$$a = b\sqrt{15} \tag{3}$$

We now consider the following set:

$$S = \{k\sqrt{15} \mid k \text{ and } k\sqrt{15} \text{ are positive integers}\}$$

Note that S is nonempty, since b and $b\sqrt{15}$ are positive integers by (2) and (3) and therefore $b\sqrt{15}$ is a member of S. Hence, by the Well-Ordering Property, S has a smallest element, say,

$$s = t\sqrt{15}, \quad \text{where } s \text{ and } t \text{ are positive integers} \tag{4}$$

Consider the following number

$$w = s\sqrt{15} - 3s$$

The goal is to show that w is from S and is smaller than s (this gives us a contradiction). To this end, we note that

(i) $s\sqrt{15}$ is a positive integer, since

$$s\sqrt{15} \stackrel{(4)}{=} t\sqrt{15} \cdot \sqrt{15} = 15t$$

which is a positive integer, since t is a positive integer by (4).

(ii) w is an integer, since $3s$ is an integer by (4), $s\sqrt{15}$ is an integer by (i) and w is the difference of these two integers. Moreover, it is a *positive* integer, since

$$w = s\sqrt{15} - 3s = s(\sqrt{15} - 3)$$

which is > 0, because $s > 0$ by (4) and $\sqrt{15} - 3 > 0$.

(iii) w is a member of S. To show that, we first observe that

$$w = s\sqrt{15} - 3s \stackrel{(4)}{=} s\sqrt{15} - 3t\sqrt{15} = (s - 3t)\sqrt{15}$$

therefore w is of the form $k\sqrt{15}$ with $k = s - 3t$. Moreover, $k\sqrt{15}$ is a positive integer, since w is a positive integer by (ii) and $k\sqrt{15} = w$.

(iv) $w < s$, since

$$w = s\sqrt{15} - 3s = s(\sqrt{15} - 3) < s$$

So, it immediately follows from (iii) and (iv) that w is an element of S which is smaller than s. This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{15}$ is irrational. ■

REMARK: This problem was given as a Midterm Exam question in Summer of 2017.