Section 7.3 Applications of Linear Programming

EXAMPLE: A 4-H Club member raises goats and pigs. She wants to raise no more than 16 animals, including no more than 10 goats. She spends \$25 to raise a goat and \$75 to raise a pig, and she has \$900 available for the project. Find the maximum profit she can make if each goat produces a profit of \$14 and each pig a profit of \$40.

Solution: The total profit is determined by the number of goats and pigs. So let x be the number of goats to be produced and let y be the number of pigs. Then summarize the information of the problem in a table.

	Number	Cost to Raise	Profit Each
Goats	x	\$25	\$14
Pigs	y	\$75	\$40
Maximum Available	16	\$900	

Use this table to write the necessary constraints. Since the total number of animals cannot exceed 16, the first constraint is

$$x + y \leq 16$$

"No more than 10 goats" means that

$$x \leq 10$$

The cost to raise x goats at \$25 per goat is $25x$ dollars, while the cost for y pigs at \$75 each is $75y$ dollars. Only \$900 is available, so

$$25x + 75y \leq 900 \quad \Longrightarrow \quad x + 3y \leq 36$$

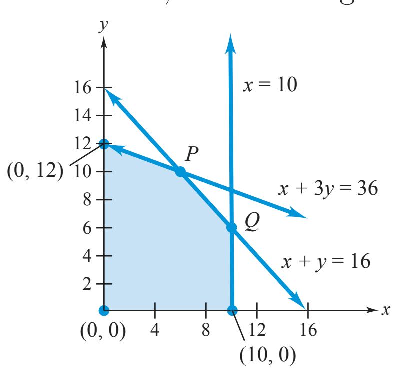
The number of goats and pigs cannot be negative, so $x \geq 0$ and $y \geq 0$. The 4-H Club member wants to know the number of goats and the number of pigs that should be raised for maximum profit. Each goat produces a profit of \$14, and each pig produces a profit of \$40. If z represents total profit, then

$$z = 14x + 40y$$

is the objective function that is to be maximized. We must solve the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & z = 14x + 40y \\ \text{subject to} \quad & x + y \leq 16 \\ & x \leq 10 \\ & x + 3y \leq 36 \\ & x \geq 0, y \geq 0 \end{aligned}$$

Using the methods of the previous section, graph the feasible region for the system of inequalities given by the constraints, as in the Figure below.



Corner Point	$z = 14x + 40y$
(0, 12)	14(0) + 40(12) = 480
(6, 10)	**14(6) + 40(10) = 484 (maximum)**
(10, 6)	14(10) + 40(6) = 380
(10, 0)	14(10) + 40(0) = 140
(0, 0)	14(0) + 40(0) = 0

Find the coordinates of the other corner points by solving a system of equations. Then test each corner point in the objective function to find the maximum profit. The maximum value for z of 484 occurs at $(6, 10)$. Thus, 6 goats and 10 pigs will produce a maximum profit of \$484.

EXAMPLE: An office manager needs to purchase new filing cabinets. He knows that Ace cabinets cost \$40 each, require 6 square feet of floor space, and hold 8 cubic feet of files. On the other hand, each Excello cabinet costs \$80, requires 8 square feet of floor space, and holds 12 cubic feet. His budget permits him to spend no more than \$560 on files, while the office has room for no more than 72 square feet of cabinets. The manager desires the greatest storage capacity within the limitations imposed by funds and space. How many of each type of cabinet should he buy?

Solution: Let x represent the number of Ace cabinets to be bought, and let y represent the number of Excello cabinets. The information given in the problem can be summarized as follows.

	Number	Cost of Each	Space Required	Storage Capacity
Ace	x	\$40	6 sq ft	8 cu ft
Excello	y	\$80	8 sq ft	12 cu ft
Maximum Available		\$560	72 sq ft	

The constraints imposed by cost and space are

$$40x + 80y \leq 560 \quad \text{Cost}$$
$$6x + 8y \leq 72. \quad \text{Floor space}$$

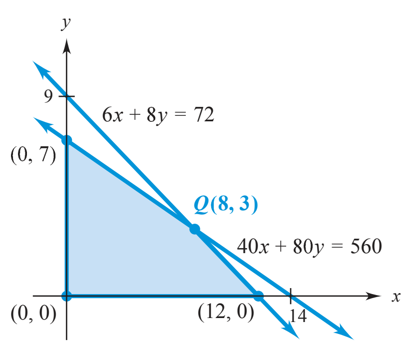
The number of cabinets cannot be negative, so $x \geq 0$ and $y \geq 0$. The objective function to be maximized gives the amount of storage capacity provided by some combination of Ace and Excello cabinets. From the information in the c hart, the objective function is

$$z = \text{ Storage capacity } = 8x + 12y$$

In sum, the given problem has produced the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & z = 8x + 12y \\ \text{subject to} \quad & 40x + 80y \leq 560 \\ & 6x + 8y \leq 72 \\ & x \geq 0, y \geq 0 \end{aligned}$$

A graph of the feasible region is shown in the Figure below. Three of the corner points can be identified from the graph as $(0, 0)$, $(0, 7)$, and $(12, 0)$. The fourth corner point, labeled Q in the figure, can be found algebraically to be (8, 3).



Corner Point	Value of $z = 8x + 12y$
(0, 0)	0
(0, 7)	84
(8, 3)	**100 (maximum)**
(12, 0)	96

Use the corner point theorem to find the maximum value of z. The objective function, which represents storage space, is maximized when $x = 8$ and $y = 3$. The manager should buy 8 Ace cabinets and 3 Excello cabinets.

EXAMPLE: Certain laboratory animals must have at least 30 grams of protein and at least 20 grams of fat per feeding period. These nutrients come from food A, which costs 18¢ per unit and supplies 2 grams of protein and 4 of fat, and food B, with 6 grams of protein and 2 of fat, costing 12¢ per unit. Food B is bought under a long-term contract requiring that at least 2 units of B be used per serving. How much of each food must be bought to produce the minimum cost per serving?

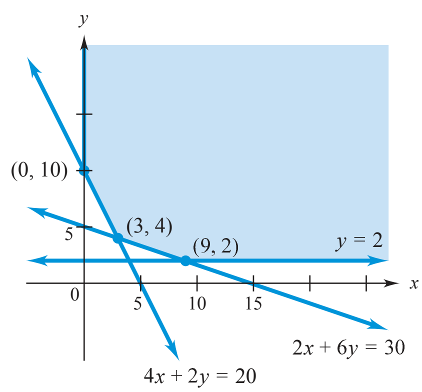
Solution: Let x represent the amount of food A needed and y the amount of food B. Use the given information to produce the following table.

Food	Number of Units	Grams of Protein	Grams of Fat	Cost
A	x	2	4	18¢
B	y	6	2	12¢
Minimum Required		30	20	

Use the table to develop the linear programming problem. Since the animals must have *at least* 30 grams of protein and 20 grams of fat, use $\geq$ in the constraint inequalities for protein and fat. The long-term contract provides a constraint not shown in the table, namely, $y \geq 2$. So we have the following problem:

$$\begin{aligned} \text{Minimize} \quad & z = .18x + .12y && \text{Cost} \\ \text{subject to} \quad & 2x + 6y \geq 30 && \text{Protein} \\ & 4x + 2y \geq 20 && \text{Fat} \\ & y \geq 2 && \text{Contract} \\ & x \geq 0, y \geq 0. \end{aligned}$$

(The constraint $y \geq 0$ is redundant because of the constraint $y \geq 2$.) A graph of the feasible region with the corner points identified is shown in the Figure below.



Use the corner point theorem to find the minimum value of z as shown in the table.

Corner Points	$z = .18x + .12y$	
(0, 10)	.18(0) + .12(10) = 1.20	
(3, 4)	**.18(3) + .12(4) = 1.02**	**(minimum)**
(9, 2)	.18(9) + .12(2) = 1.86	

The minimum value of 1.02 occurs at $(3, 4)$. Thus, 3 units of food A and 4 units of food B will produce a minimum cost of \$1.02 per serving.

The feasible region in the Figure above is an unbounded one: The region extends indefinitely to the upper right. With this region, it would not be possible to *maximize* the objective function, because the total cost of the food could always be increased by encouraging the animals to eat more.

One measure of the risk involved in investing in a stock or mutual fund is called the standard deviation. The standard deviation measures the volatility of investment returns relative to an historical average. If the return of an investment tool fluctuates a great deal from the historical average return, then there will be a higher standard deviation value for that stock. If an investment tool's value stays near the historical average, then it will have a small standard deviation value. Thus, a higher standard deviation for an investment tool can be one measure of higher risk. Investors often wish to obtain the highest profit while minimizing risk.

EXAMPLE: Carolyn Behr-Jerome wants to invest up to $5000 in stocks. The share price for the Costco Whole Corporation (COST) is $108, and it has a standard deviation value of 15.9. The share price for CVS Caremark Corporation (CVS) is $59, and the standard deviation value is 19.5. Based on the average of 10-year returns, Costco would produce in a year a profit of $15 per share and CVS would produce a profit of $10 a share. Carolyn would like to obtain at least $800 in profit. How many shares of each stock should she purchase to minimize the risk as measured by the standard deviation? What is the minimum value of the risk? (Data from: www.morningstar.com and www.abg-analytics.com as of April 2013.)

Solution: Let x represent the number of shares of Costco stock to be purchased, and let y be the number of shares of CVS stock to be purchased. The information in the problem can be summarized as follows:

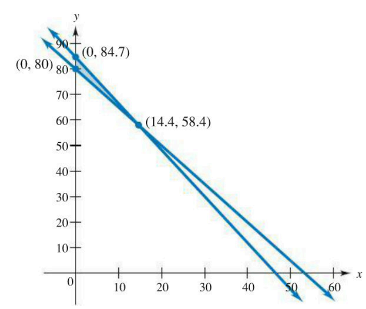
	Number of Shares	Cost of Each	Profit	Risk
Costco	x	$108	$15	15.9
CVS	y	$59	$10	19.5
Constraints		$5000	$800	

The constraints imposed by the cost of the shares and the profits are

$$108x + 59y \leq 5000$$
$$15x + 10y \geq 800$$

The number of stocks to be purchased cannot be negative, so $x \geq 0$ and $y \geq 0$. The objective function to be minimized gives the amount of risk provided by some combination of shares in Costco and CVS stocks. From the infom1ation in the chart, the objective function is

$$\begin{aligned} \text{Minimize} \quad & z = 15.9x + 19.5y \\ \text{subject to} \quad & 108x + 59y \leq 5000 \\ & 15x + 10y \geq 800 \\ & x \geq 0, y \geq 0 \end{aligned}$$



A graph of the feasible region with the corner points identified is shown in the Figure above.

Corner Points	$z = 15.9x + 19.5y$
(0, 84.7)	15.9(0) + 19.5(84.7) = 1651.65
(0, 80)	15.9(0) + 19.5(80) = 1560.0
(14.4, 58.4)	15.9(14.4) + 19.5(58.4) = 1367.76

Use the corner point theorem to find the minimum value of z as shown in the table above. The minimum value of 1367.76 occurs at $(14.4, 58.4)$. Since Carolyn must buy a whole share of stock, she would buy 14 shares of Costco stock and 58 shares of CVS stock.