

Section 7.2 Linear Programming: The Graphical Method

Many problems in business, science, and economics involve finding the optimal value of a function (for instance, the maximum value of the profit function or the minimum value of the cost function), subject to various **constraints** (such as transportation costs, environmental protection laws, availability of parts, and interest rates). **Linear programming** deals with such situations. In linear programming, the function to be optimized, called the **objective function**, is linear and the constraints are given by linear inequalities. Linear programming problems that involve only two variables can be solved by the graphical method, explained in the Example below.

EXAMPLE: Find the maximum and minimum values of the objective function $z = 2x + 5y$, subject to the following constraints:

$$\begin{aligned}3x + 2y &\leq 6 \\ -2x + 4y &\leq 8 \\ x + y &\geq 1 \\ x &\geq 0, y \geq 0\end{aligned}$$

Solution: First, graph the feasible region of the system of inequalities (see the Figure below (left)). The points in this region or on its boundaries are the only ones that satisfy all the constraints. However, each such point may produce a different value of the objective function. For instance, the points $(.5, 1)$ and $(1, 0)$ in the feasible region lead to the respective values

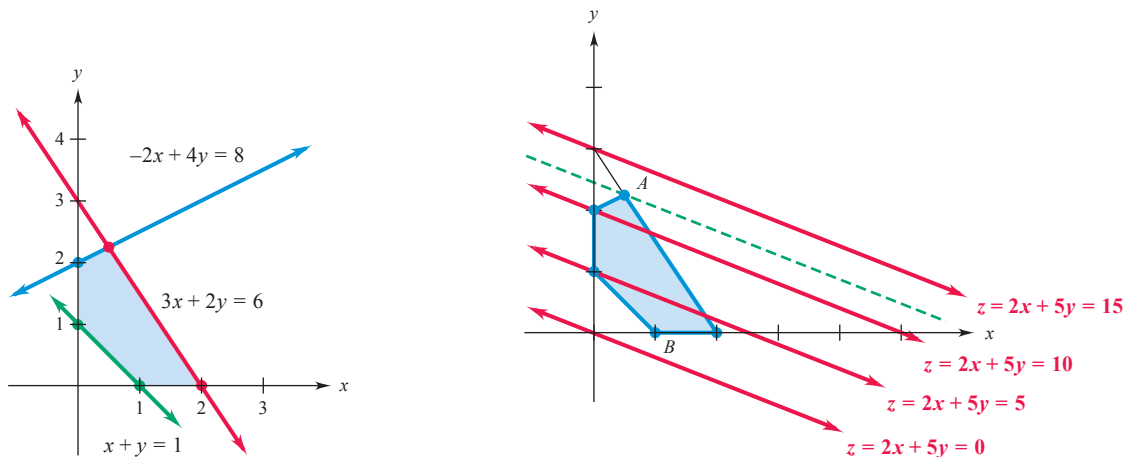
$$z = 2(.5) + 5(1) = 6 \quad \text{and} \quad z = 2(1) + 5(0) = 2$$

We must find the points that produce the maximum and minimum values of z .

To find the maximum value, consider various possible values for z . For instance, when $z = 0$, the objective function is $0 = 2x + 5y$, whose graph is a straight line. Similarly, when z is 5, 10, and 15, the objective function becomes (in turn)

$$5 = 2x + 5y, \quad 10 = 2x + 5y, \quad \text{and} \quad 15 = 2x + 5y$$

These four lines are graphed in the Figure below (right). (All the lines are parallel because they have the same slope.) The figure shows that z cannot take on the value 15, because the graph for $z = 15$ is entirely outside the feasible region. The maximum possible value of z will be obtained from a line parallel to the others and between the lines representing the objective



function when $z = 10$ and $z = 15$. The value of z will be as large as possible, and all constraints will be satisfied, if this line just touches the feasible region. This occurs with the green line through point A.

To find coordinates of the intersection point A of the graphs of $3x + 2y = 6$ and $-2x + 4y = 8$ (see the Figure above (left)), we use the elimination method:

$$\begin{cases} 3x + 2y = 6 \\ -2x + 4y = 8 \end{cases} \iff \begin{cases} 6x + 4y = 12 \\ -6x + 12y = 24 \end{cases} \implies 16y = 36 \iff y = \frac{36}{16} = \frac{9}{4}$$

and

$$\begin{cases} 3x + 2y = 6 \\ -2x + 4y = 8 \end{cases} \iff \begin{cases} 6x + 4y = 12 \\ -2x + 4y = 8 \end{cases} \implies 8x = 4 \iff x = \frac{4}{8} = \frac{1}{2}$$

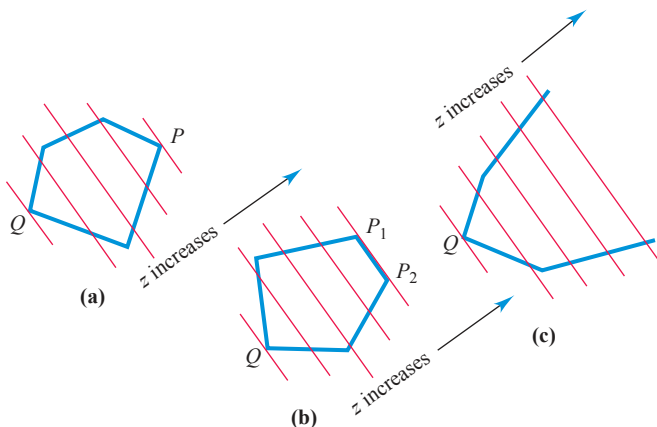
The value of z at point A is

$$z = 2x + 5y = 2(.5) + 5(2.25) = 12.25$$

Thus, the maximum possible value of z is 12.25. Similarly, the minimum value of z occurs at point B , which has coordinates $(1, 0)$. The minimum value of z is $2(1) + 5(0) = 2$.

Points such as A and B in the Example above are called corner points. A **corner point** is a point in the feasible region where the boundary lines of two constraints cross. The feasible region in the Figure above (left) is **bounded** because the region is enclosed by boundary lines on all sides. Linear programming problems with bounded regions always have solutions. However, if the Example above did not include the constraint $3x + 2y \leq 6$, the feasible region would be **unbounded**, and there would be no way to *maximize* the value of the objective function.

Some general conclusions can be drawn from the method of solution used in the Example above. The Figure below shows various feasible regions and the lines that result from various values of z . (The Figure shows the situation in which the lines are in order from left to right as z increases.) In part (a) of the Figure, the objective function takes on its minimum value at corner point Q and its maximum value at corner point P . The minimum is again at Q in part (b), but the maximum occurs at P_1 or P_2 , or any point on the line segment connecting them. Finally, in part (c), the minimum value occurs at Q , but the objective function has no maximum value because the feasible region is unbounded.



The preceding discussion suggests the **corner point theorem**.

Corner Point Theorem

If the feasible region is bounded, then the objective function has both a maximum and a minimum value, and each occurs at one or more corner points.

If the feasible region is unbounded, the objective function may not have a maximum or minimum. But if a maximum or minimum value exists, it will occur at one or more corner points.

This theorem simplifies the job of finding an optimum value. First, graph the feasible region and find all corner points. Then test each point in the objective function. Finally, identify the corner point producing the optimum solution.

With the theorem, the problem in the Example above could have been solved by identifying the five corner points: $(0, 1)$, $(0, 2)$, $(.5, 2.25)$, $(2, 0)$, and $(1, 0)$. Then, substituting each of these points into the objective function $z = 2x + 5y$ would identify the corner points that produce the maximum and minimum values of z .

Corner Point	Value of $z = 2x + 5y$
$(0, 1)$	$2(0) + 5(1) = 5$
$(0, 2)$	$2(0) + 5(2) = 10$
$(.5, 2.25)$	$2(.5) + 5(2.25) = 12.25$ (maximum)
$(2, 0)$	$2(2) + 5(0) = 4$
$(1, 0)$	$2(1) + 5(0) = 2$ (minimum)

From these results, the corner point $(.5, 2.25)$ yields the maximum value of 12.25 and the corner point $(1, 0)$ gives the minimum value of 2. These are the same values found earlier.

A summary of the steps for solving a linear programming problem by the graphical method is given here.

Solving a Linear Programming Problem Graphically

1. Write the objective function and all necessary constraints.
2. Graph the feasible region.
3. Determine the coordinates of each of the corner points.
4. Find the value of the objective function at each corner point.
5. If the feasible region is bounded, the solution is given by the corner point producing the optimum value of the objective function.
6. If the feasible region is an unbounded region in the first quadrant and both coefficients of the objective function are positive,* then the minimum value of the objective function occurs at a corner point and there is no maximum value.

EXAMPLE: Sketch the feasible region for the following set of constraints:

$$\begin{aligned} 3y - 2x &\geq 0 \\ y + 8x &\leq 52 \\ y - 2x &\leq 2 \\ x &\geq 3 \end{aligned}$$

Then find the maximum and minimum values of the objective function $z = 5x + 2y$.

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Then find the maximum and minimum values of the objective function $z = 5x + 2y$.

Solution: Graph the feasible region, as in the Figure below. To find the corner points, you must solve these four systems of equations:

A	B	C	D
$y - 2x = 2$	$3y - 2x = 0$	$3y - 2x = 0$	$y - 2x = 2$
$x = 3$	$x = 3$	$y + 8x = 52$	$y + 8x = 52$

The first two systems are easily solved by substitution, which shows that $A = (3, 8)$ and $B = (3, 2)$. We now solve the other two systems:

System C:

$$\begin{cases} 3y - 2x = 0 \\ y + 8x = 52 \end{cases} \iff \begin{cases} 3y - 2x = 0 \\ 3y + 24x = 156 \end{cases} \implies 26x = 156 \iff x = \frac{156}{26} = 6$$

and

$$\begin{cases} 3y - 2x = 0 \\ y + 8x = 52 \end{cases} \iff \begin{cases} 12y - 8x = 0 \\ y + 8x = 52 \end{cases} \implies 13y = 52 \iff y = \frac{52}{13} = 4$$

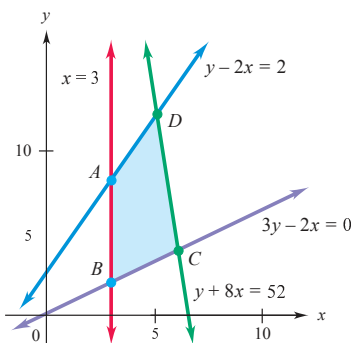
System D:

$$\begin{cases} y - 2x = 2 \\ y + 8x = 52 \end{cases} \implies 10x = 50 \iff x = \frac{50}{10} = 5$$

and

$$\begin{cases} y - 2x = 2 \\ y + 8x = 52 \end{cases} \iff \begin{cases} 4y - 8x = 8 \\ y + 8x = 52 \end{cases} \implies 5y = 60 \iff y = \frac{60}{5} = 12$$

Hence, $C = (6, 4)$ and $D = (5, 12)$.



Use the corner points from the graph to find the maximum and minimum values of the objective function.

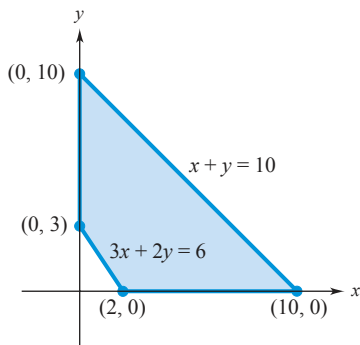
Corner Point	Value of $z = 5x + 2y$
(3, 8)	$5(3) + 2(8) = 31$
(3, 2)	$5(3) + 2(2) = 19$ (minimum)
(6, 4)	$5(6) + 2(4) = 38$
(5, 12)	$5(5) + 2(12) = 49$ (maximum)

The minimum value of $z = 5x + 2y$ is 19, at the corner point (3, 2). The maximum value is 49, at (5, 12).

EXAMPLE: Solve the following linear programming problem:

$$\begin{aligned} &\text{Minimize } z = x + 2y \\ &\text{subject to } x + y \leq 10 \\ &\quad\quad\quad 3x + 2y \geq 6 \\ &\quad\quad\quad x \geq 0, y \geq 0 \end{aligned}$$

Solution: The feasible region is shown in the Figure below (left). From the figure, the corner points are $(0, 3)$, $(0, 10)$, $(10, 0)$, and $(2, 0)$. These corner points give the following values of z .



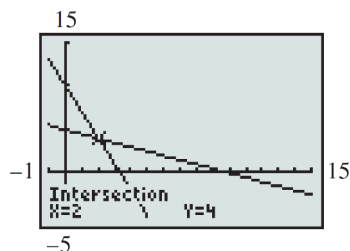
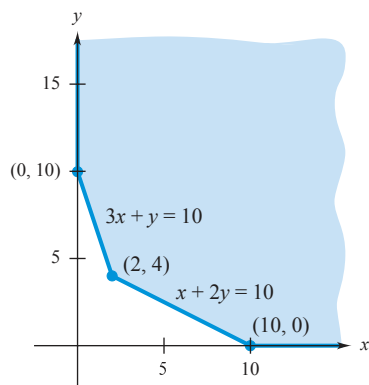
Corner Point	Value of $z = x + 2y$
$(0, 3)$	$0 + 2(3) = 6$
$(0, 10)$	$0 + 2(10) = 20$
$(10, 0)$	$10 + 2(0) = 10$
$(2, 0)$	$2 + 2(0) = 2$ (minimum)

The minimum value of z is 2; it occurs at $(2, 0)$.

EXAMPLE: Solve the following linear programming problem:

$$\begin{aligned} &\text{Minimize } z = 2x + 4y \\ &\text{subject to } x + 2y \geq 10 \\ &\quad\quad\quad 3x + y \geq 10 \\ &\quad\quad\quad x \geq 0, y \geq 0 \end{aligned}$$

Solution: The Figure below shows the hand-drawn graph with corner points $(0, 10)$, $(2, 4)$, and $(10, 0)$, as well as the calculator graph with the corner point $(2, 4)$. Find the value of z for each point.



In this case, both $(2, 4)$ and $(10, 0)$, as well as all the points on the boundary line between them, give the same optimum value of z . So there is an infinite number of equally “good” values of x and y that give the same minimum value of the objective function $z = 2x + 4y$. The minimum value is 20.

Corner Point	Value of $z = 2x + 4y$
$(0, 10)$	$2(0) + 4(10) = 40$
$(2, 4)$	$2(2) + 4(4) = 20$ (minimum)
$(10, 0)$	$2(10) + 4(0) = 20$ (minimum)