Section 7.1 Graphing Linear Inequalities in Two Variables

Examples of linear inequalities in two variables include

$$x + 2y < 4, \quad 3x + 2y > 6, \quad \text{and} \quad 2x - 5y \geq 10$$

A solution of a linear inequality is an ordered pair that satisfies the inequality. For example $(4, 4)$ is a solution of

$$3x - 2y \leq 6$$

(Check by substituting 4 for x and 4 for y.) A linear inequality has an infinite number of solutions, one for every choice of a value for x. The best way to show these solutions is to sketch the **graph of the inequality**, which consists of all points in the plane whose coordinates satisfy the inequality.

EXAMPLE: Graph the inequality $3x - 2y \leq 6$.

Solution: First, solve the inequality for y:

$$3x - 2y \leq 6$$

$$-2y \leq -3x + 6$$

$$\frac{-2y}{-2} \geq \frac{-3x+6}{-2} \quad \Longleftrightarrow \quad y \geq \frac{-3x}{-2} + \frac{6}{-2} = 1.5x - 3$$

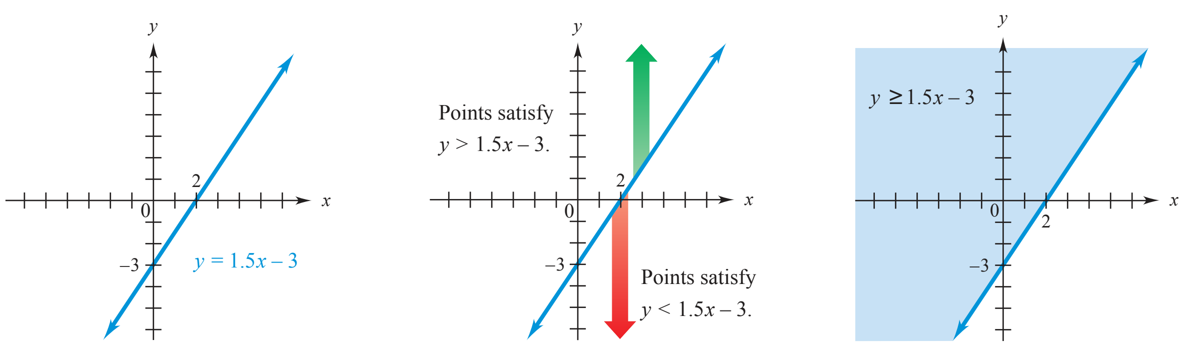
This inequality has the same solutions as the original one. To solve it, note that the points on the line $y = 1.5x - 3$ certainly satisfy $y \geq 1.5x - 3$. Plot some points, and graph this line, as in the Figure below (left).

The points on the line satisfy "y *equals* $1.5x - 3$." The points satisfying "y *is greater than* $1.5x - 3$" are the points *above* the line (because they have larger second coordinates than the points on the line; see the Figure below (middle)). Similarly, the points satisfying

$$y < 1.5x - 3$$

lie below the line (because they have smaller second coordinates), as shown in the Figure below (middle). The line $y = 1.5x - 3$ is the **boundary line**.

Thus, the solutions of $y \geq 1.5x - 3$ are all points *on or above* the line $y = 1.5x - 3$. The line and the shaded region of the Figure below (right) make up the graph of the inequality $y \geq 1.5x - 3$.



EXAMPLE: Graph $x + 4y < 4$.

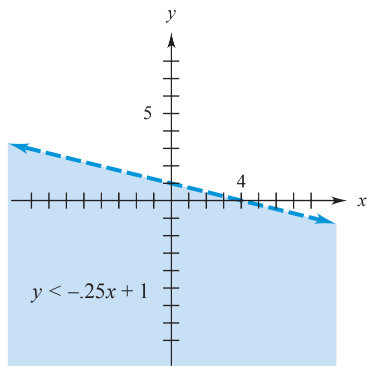
EXAMPLE: Graph $x + 4y < 4$.

Solution: First obtain an equivalent inequality by solving for y:

$$4y < -x + 4$$

$$\frac{4y}{4} < \frac{-x+4}{4} \quad \Longleftrightarrow \quad y < \frac{-x}{4} + \frac{4}{4} = -.25x + 1$$

The boundary line is $y = -.25x + 1$, but it is *not* part of the solution, since points *on* the line do not satisfy $y < -.25x + 1$. To indicate this, the line is drawn dashed in the Figure below. The points *below* the boundary line are the solutions of $y < -.25x + 1$, because they have smaller second coordinates than the points on the line $y = -.25x + 1$. The shaded region in the Figure below (excluding the dashed line) is the graph of the inequality $y < -.25x + 1$.



The Examples above show that the solutions of a linear inequality form a **half-plane** consisting of all points on one side of the boundary line (and possibly the line itself). When an inequality is solved for y, the inequality symbol immediately tells whether the points above (>), on (=), or below (<) the boundary line satisfy the inequality, as summarized here.

Inequality	Solution Consists of All Points
$y \geq mx + b$	***on or above*** the line $y = mx + b$
$y > mx + b$	***above*** the line $y = mx + b$
$y \leq mx + b$	***on or below*** the line $y = mx + b$
$y < mx + b$	***below*** the line $y = mx + b$

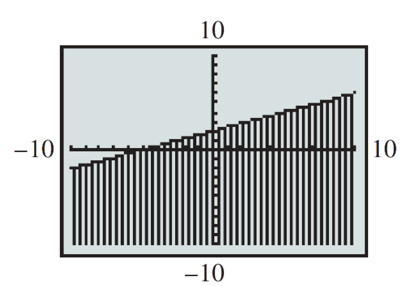
When graphing by hand, draw the boundary line $y = mx + b$ solid when it is included in the solution ($\geq$ or $\leq$ inequalities) and dashed when it is not part of the solution ($>$ or $<$ inequalities).

EXAMPLE: Graph $5y - 2x \leq 10$.

Solution: Solve the inequality for y:

$$5y \leq 2x + 10$$

$$\frac{5y}{5} \leq \frac{2x+10}{5} \quad \Longleftrightarrow \quad y \leq \frac{2x}{5} + \frac{10}{5} = .4x + 2$$



The graph consists of all points on or below the boundary line $y \leq .4x+2$, as shown in the Figure on the right.

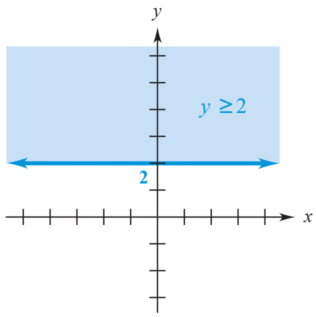
EXAMPLE: Graph each of the given inequalities.

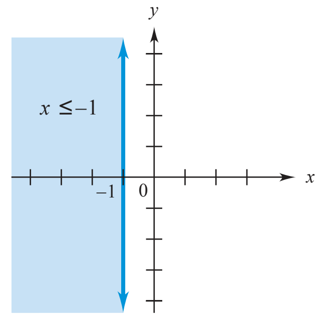
(a) $y \geq 2$

Solution: The boundary line is the horizontal line $y = 2$. The graph consists of all points on or above this line (see the Figure below (left)).

(b) $x \leq -1$

Solution: This inequality does not fit the pattern discussed earlier, but it can be solved by a similar technique. Here, the boundary line is the vertical line $x = -1$, and it is included in the solution. The points satisfying $x < -1$ are all points to the left of this line (because they have x-coordinates smaller than -1). So the graph consists of the points that are *on or to the left* of the vertical line $x = -1$, as shown in the Figure below (right).





An alternative technique for solving inequalities that does not require solving for y is illustrated in the next example.

EXAMPLE: Graph $4y - 2x \geq 6$.

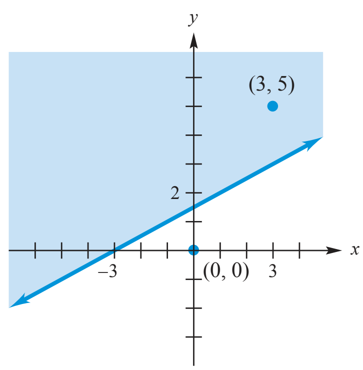
Solution: The boundary line is $4y - 2x = 6$, which can be graphed by plotting its x- and y-intercepts. The graph contains the half-plane above or below this line. To determine which, choose a test point — any point not on the boundary line, say, $(0, 0)$. Letting $x = 0$ and $y = 0$ in the inequality produces

$$4(0) - 2(0) \geq 6, \quad \text{a } \textit{false} \text{ statement.}$$

Therefore, $(0, 0)$ is not in the solution. So the solution is the half-plane that does *not* include $(0, 0)$, as shown in the Figure below. If a different test point is used, say, $(3, 5)$, then substituting $x = 3$ and $y = 5$ in the inequality produces

$$4(5) - 2(3) \geq 6, \quad \text{a } \textit{true} \text{ statement.}$$

Therefore, the solution of the inequality is the half-plane containing $(3, 5)$, as shown in the Figure below.



Systems of Inequalities

Real-world problems often involve many inequalities. For example, a manufacturing problem might produce inequalities resulting from production requirements, as well as inequalities about cost requirements. A set of at least two inequalities is called a **system of inequalities**. The **graph** of a system of inequalities is made up of all those points which satisfy *all* the inequalities of the system.

EXAMPLE: Graph the system

$$3x + y \le 12$$
$$x \le 2y$$

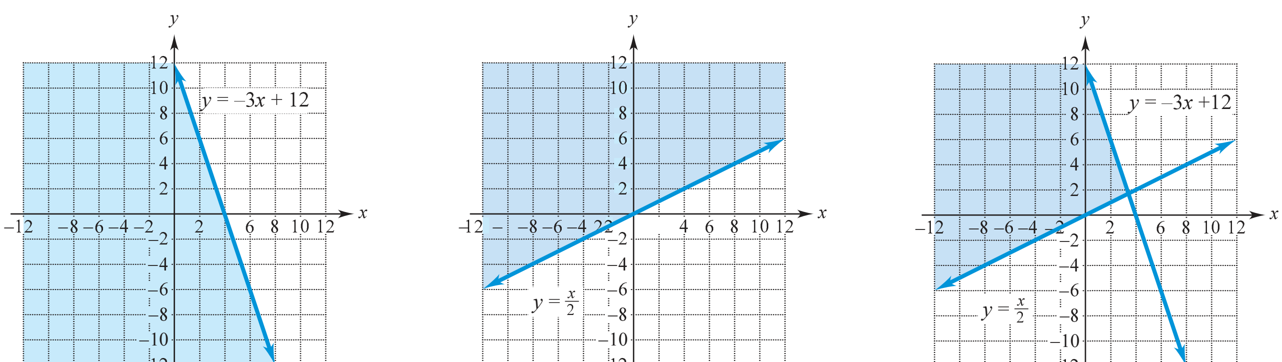
Solution: First, solve each inequality for y:

$$3x + y \le 12 \qquad\qquad x \le 2y$$
$$y \le -3x + 12 \qquad\qquad y \ge \frac{x}{2}$$

Then the original system is equivalent to this one:

$$y \le -3x + 12$$
$$y \ge \frac{x}{2}$$

The solutions of the first inequality are the points *on or below* the line $y = -3x + 12$ (see the Figure below (left)). The solutions of the second inequality are the points *on or above* the line $y = x/2$ (see the Figure below (middle)). So the solutions of the system are the points that satisfy both of these conditions, as shown in the Figure below (right).



The shaded region in the Figure above (right) is sometimes called the **region of feasible solutions**, or just the **feasible region**, since it consists of all the points that satisfy (are feasible for) every inequality of the system.

EXAMPLE: Graph the feasible region for the system

$$2x - 5y \le 10$$
$$x + 2y \le 8$$
$$x \ge 0, \; y \ge 0$$

EXAMPLE: Graph the feasible region for the system

$$2x - 5y \le 10$$
$$x + 2y \le 8$$
$$x \ge 0, \; y \ge 0$$

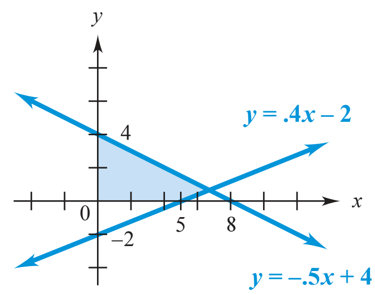
Solution: Begin by solving the first two inequalities for y:

$$2x - 5y \le 10 \qquad\qquad x + 2y \le 8$$
$$-5y \le -2x + 10 \qquad\qquad 2y \le -x + 8$$
$$y \ge .4x - 2 \qquad\qquad y \le -.5x + 4$$

Then the original system is equivalent to this one:

$$y \ge .4x - 2$$
$$y \le -.5x + 4$$
$$x \ge 0, \; y \ge 0$$

The inequalities $x \ge 0$ and $y \ge 0$ restrict the graph to the first quadrant. So the feasible region consists of all points in the first quadrant that are on or above the line $y = .4x - 2$ and on or below the line $y = -.5x + 4$ (see the Figure on the right).



EXAMPLE: Graph the feasible region for the system

$$5y - 2x \le 10$$
$$x \ge 3, \; y \ge 2$$

Solution: Solve the first inequality for y:

$$5y - 2x \le 10$$
$$5y \le 2x + 10$$
$$\frac{5y}{5} \le \frac{2x+10}{5} \iff y \le \frac{2x}{5} + \frac{10}{5} = .4x + 2$$

Then the original system is equivalent to this one:

$$y \le .4x + 2$$
$$x \ge 3, \; y \ge 2$$

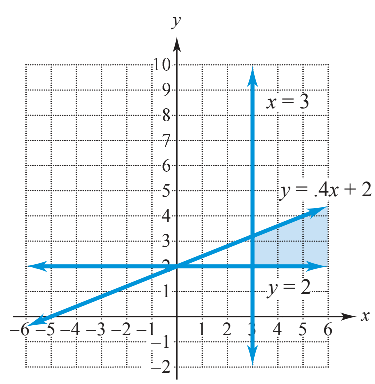
One can show that the feasible region consists of all points that lie

on or be low the line $y \le .4x + 2$ *and*

on or to the right of the vertical line $x = 3$ *and*

on or above the horizontal line $y = 2$

as shown in the Figure on the right.



EXAMPLE: Midtown Manufacturing Company makes plastic plates and cups, both of which require time on two machines. Producing a unit of plates requires 1 hour on machine A and 2 on machine B, while producing a unit of cups requires 3 hours on machine A and 1 on machine B. Each machine is operated for at most 15 hours per day. Write a system of inequalities expressing these conditions, and graph the feasible region.

Solution: Let x represent the number of units of plates to be made and y represent the number of units of cups. Then make a chart that summarizes the given information.

		Time on Machine	
	Number of Units	***A***	***B***
Plates	x	1	2
Cups	y	3	1
Maximum Time Available		15	15

We must have $x \geq 0$ and $y \geq 0$ because the company cannot produce a negative number of cups or plates. On machine A, producing x units of plates requires a total of $1 \cdot x = x$ hours, while producing y units of cups requires $3 \cdot y = 3y$ hours. Since machine A is available no more than 15 hours a day,

$$x + 3y \leq 15$$

$$y \leq -\frac{x}{3} + 5$$

Similarly, the requirement that machine B be used no more than 15 hours a day gives

$$2x + y \leq 15$$

$$y \leq -2x + 15$$

So we must solve the system

$$y \leq -\frac{x}{3} + 5$$

$$y \leq -2y + 15$$

$$x \geq 0, \ y \geq 0$$

The feasible region is shown in the Figure below.