Section 6.3 Applications of Systems of Linear Equations

EXAMPLE: A small business invests \$10,000 in equipment to produce a new soft drink. Each bottle of the soft drink costs \$0.65 to produce and is sold for \$1.20. How many bottles must be sold before the business breaks even?

Solution: The total cost of producing x bottles is

$$\text{Total cost} = \text{Cost per bottle} \cdot \text{Number of bottles} + \text{Initial cost}$$

$$C = 0.65x + 10{,}000. \qquad \text{Equation 1}$$

The revenue obtained by selling x bottles is

$$\text{Total revenue} = \text{Price per bottle} \cdot \text{Number of bottles}$$

$$R = 1.20x. \qquad \text{Equation 2}$$

Because the break-even point occurs when $R = C$, you have $C = 1.20x$, and the system of equations to solve is

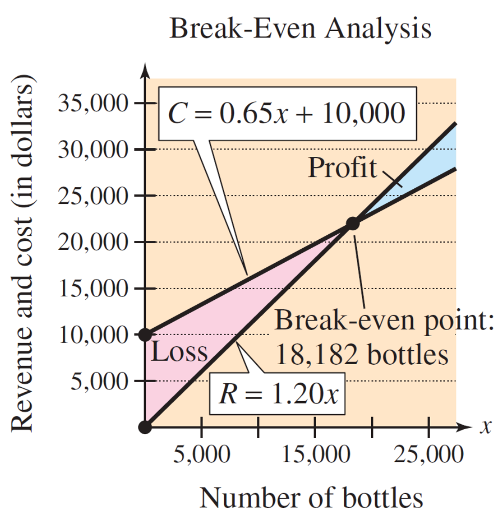
$$\begin{cases} C = 0.65x + 10{,}000 \\ C = 1.20x \end{cases}$$

Now we can solve by substitution.

$$1.20x = 0.65x + 10{,}000 \qquad \text{Substitute } 1.20x \text{ for } C \text{ in Equation 1.}$$

$$0.55x = 10{,}000 \qquad \text{Subtract } 0.65x \text{ from each side.}$$

$$x = \frac{10{,}000}{0.55} \approx 18{,}182 \text{ bottles.} \qquad \text{Divide each side by 0.55.}$$



EXAMPLE: The demand and supply equations for a new type of personal digital assistant are

$$\begin{cases} p = 150 - 0.00001x & \text{Demand equation} \\ p = 60 + 0.00002x & \text{Supply equation} \end{cases}$$

where p is the price in dollars and x represents the number of units. Find the equilibrium point for this market.

REMARK: Recall that the equilibrium point is the price p and number of units x that satisfy both the demand and supply equations.

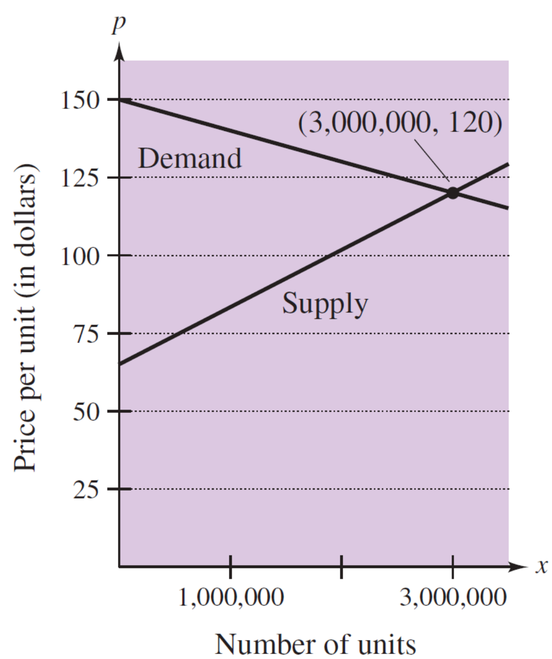
Solution: Because p is written in terms of x, begin by substituting the value of p given in the supply equation into the demand equation.

$$\begin{aligned} p &= 150 - 0.00001x && \text{Write demand equation.} \\ 60 + 0.00002x &= 150 - 0.00001x && \text{Substitute } 60 + 0.00002x \text{ for } p. \\ 0.00003x &= 90 && \text{Combine like terms.} \\ x &= 3{,}000{,}000 && \text{Solve for } x. \end{aligned}$$

So, the equilibrium point occurs when the demand and supply are each 3 million units. (See the Figure below.) The price that corresponds to this x-value is obtained by back-substituting $x = 3,000,000$ into either of the original equations. For instance, back-substituting into the demand equation produces

$$p = 150 - 0.00001(3{,}000{,}000) = 150 - 30 = \$120.$$

The solution is $(3,000,000, 120)$. Check this in both equations in the original system.



EXAMPLE: Eight hundred people attend a basketball game, and total ticket sales are \$3102. If adult tickets are \$6 and student tickets are \$3, how many adults and how many students attended the game?

Solution: Let x be the number of adults and y the number of students. Then

$$\text{Number of adults} + \text{Number of students} = \text{Total attendance.}$$

$$x + y = 800.$$

A second equation can be found by considering ticket sales:

$$\underbrace{\begin{pmatrix}\text{Price}\\ \text{per}\\ \text{ticket}\end{pmatrix} \times \begin{pmatrix}\text{Number}\\ \text{of}\\ \text{adults}\end{pmatrix}}_{\text{Adult ticket sales}} + \underbrace{\begin{pmatrix}\text{Price}\\ \text{per}\\ \text{ticket}\end{pmatrix} \times \begin{pmatrix}\text{Number}\\ \text{of}\\ \text{students}\end{pmatrix}}_{\text{Student ticket sales}} = 3102 \quad (= \text{Total ticket sales})$$

$$6x + 3y = 3102.$$

To find x and y we must solve this system of equations:

$$\begin{cases} x + y = 800 \\ 6x + 3y = 3102 \end{cases}$$

Form the augmented matrix and transform it into *reduced* row-echelon form:

$$\left[\begin{array}{cc|c} 1 & 1 & 800 \\ 6 & 3 & 3102 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 800 \\ 0 & -3 & -1698 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 800 \\ 0 & 1 & 566 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 234 \\ 0 & 1 & 566 \end{array}\right]$$

It follows that $x = 234$ and $y = 566$. This shows that 234 adults and 566 students attended the game.

EXAMPLE: The U-Drive Rent-a-Truck Company plans to spend \$5 million on 200 new vehicles. Each van will cost \$20,000, each small truck \$25,000, and each large truck \$35,000. Past experience shows that U-Drive needs twice as many vans as small trucks. How many of each kind of vehicle can the company buy?

Solution: Let x be the number of vans, y the number of small trucks, and z the number of large trucks. Then

$$\begin{pmatrix}\text{Number of}\\ \text{vans}\end{pmatrix} + \begin{pmatrix}\text{Number of}\\ \text{small trucks}\end{pmatrix} + \begin{pmatrix}\text{Number of}\\ \text{large trucks}\end{pmatrix} = \text{Total number of vehicles}$$

$$x + y + z = 200.$$

Similarly,

$$\begin{pmatrix}\text{Cost of } x\\ \text{vans}\end{pmatrix} + \begin{pmatrix}\text{Cost of } y\\ \text{small trucks}\end{pmatrix} + \begin{pmatrix}\text{Cost of } z\\ \text{large trucks}\end{pmatrix} = \text{Total cost}$$

$$20{,}000x + 25{,}000y + 35{,}000z = 5{,}000{,}000.$$

Dividing both sides by 5000 produces the equivalent equation

$$4x + 5y + 7z = 1000$$

Finally, the number of vans is twice the number of small trucks; that is, $x = 2y$, or equivalently,

$$x - 2y = 0$$

To find x, y, and z we must solve this system of equations:

$$\begin{cases} x + y + z = 200 \\ 4x + 5y + 7z = 1000 \\ x - 2y = 0 \end{cases}$$

Form the augmented matrix and transform it into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 4 & 5 & 7 & 1000 \\ 1 & -2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 0 & 1 & 3 & 200 \\ 0 & -3 & -1 & -200 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 0 & 1 & 3 & 200 \\ 0 & 0 & 8 & 400 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 0 & 1 & 3 & 200 \\ 0 & 0 & 1 & 50 \end{array}\right]$$

This row-echelon matrix corresponds to the system

$$\begin{cases} x + y + z = 200 \\ \quad y + 3z = 200 \\ \qquad z = 50 \end{cases}$$

Use back substitution to solve this system:

$$z = 50 \qquad \begin{aligned} y + 3z &= 200 \\ y + 3(50) &= 200 \\ y + 150 &= 200 \\ y &= 50 \end{aligned} \qquad \begin{aligned} x + y + z &= 200 \\ x + 50 + 50 &= 200 \\ x + 100 &= 200 \\ x &= 100 \end{aligned}$$

Therefore, U-Drive should buy 100 vans, 50 small trucks, and 50 large trucks.

EXAMPLE: Ellen McGillicuddy plans to invest a total of \$100,000 in a money market account, a bond fund, an international stock fund, and a domestic stock fund. She wants 60% of her investment to be conservative (money market and bonds). She wants the amount in international stocks to be one-fourth of the amount in domestic stocks. Finally, she needs an annual return of \$4000. Assuming she gets annual returns of 2.5% on the money market account, 3.5% on the bond fund, 5% on the international stock fund, and 6% on the domestic stock fund, how much should she put in each investment?

Solution: Let x be the amount invested in the money market account, y the amount in the bond fund, z the amount in the international stock fund, and w the amount in the domestic stock fund. Then

$$x + y + z + w = \text{total amount invested} = 100,000$$

Use her annual return to get a second equation:

$$\begin{pmatrix}\text{2.5\% return} \\ \text{on money} \\ \text{market}\end{pmatrix} + \begin{pmatrix}\text{3.5\% return} \\ \text{on bond} \\ \text{fund}\end{pmatrix} + \begin{pmatrix}\text{5\% return on} \\ \text{international} \\ \text{stock fund}\end{pmatrix} + \begin{pmatrix}\text{6\% return} \\ \text{on domestic} \\ \text{stock fund}\end{pmatrix} = 4000$$

$$.025x \quad + \quad .035y \quad + \quad .05z \quad + \quad .06w \quad = 4000.$$

Since Ellen wants the amount in international stocks to be one-fourth of the amount in domestic stocks, we have

$$z = \frac{1}{4}w, \quad \text{or equivalently,} \quad z - .25w = 0$$

Finally, the amount in conservative investments is $x + y$, and this quantity should be equal to 60% of \$100,000 — that is,

$$x + y = 60,000$$

To find $x,\ y,\ z$, and w we must solve this system of equations:

$$\begin{cases} x+y+z+w=100,000 \\ .025x+.035y+.05z+.06w=4,000 \\ z-.25w=0 \\ x+y=60,000 \end{cases}$$

Form the augmented matrix and transform it into row-echelon form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 100{,}000 \\ .025 & .035 & .05 & .06 & 4{,}000 \\ 0 & 0 & 1 & -.25 & 0 \\ 1 & 1 & 0 & 0 & 60{,}000 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 100{,}000 \\ 25 & 35 & 50 & 60 & 4{,}000{,}000 \\ 0 & 0 & 1 & -.25 & 0 \\ 1 & 1 & 0 & 0 & 60{,}000 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 100{,}000 \\ 5 & 7 & 10 & 12 & 800{,}000 \\ 0 & 0 & 1 & -.25 & 0 \\ 1 & 1 & 0 & 0 & 60{,}000 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 100{,}000 \\ 0 & 2 & 5 & 7 & 300{,}000 \\ 0 & 0 & 1 & -.25 & 0 \\ 0 & 0 & -1 & -1 & -40,000 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 100{,}000 \\ 0 & 1 & 2.5 & 3.5 & 150{,}000 \\ 0 & 0 & 1 & -.25 & 0 \\ 0 & 0 & 0 & -1.25 & -40,000 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 100{,}000 \\ 0 & 1 & 2.5 & 3.5 & 150{,}000 \\ 0 & 0 & 1 & -.25 & 0 \\ 0 & 0 & 0 & 1 & 32{,}000 \end{array}\right]$$

This row-echelon matrix corresponds to the system

$$\begin{aligned} x+y+z+w &= 100,000 \\ y+2.5z+3.5w &= 150,000 \\ z-.25w &= 0 \\ w &= 32,000 \end{aligned}$$

Back substitution shows that

$$\begin{aligned} w &= 32,000 \\ z &= .25w = .25(32,000) = 8000 \\ y &= -2.5z-3.5w+150,000 = -2.5(8000)-3.5(32,000)+150,000 = 18,000 \\ x &= -y-z-w+100,000 = -18,000-8000-32,000+100,000 = 42,000 \end{aligned}$$

Therefore, Ellen should put \$42,000 in the money market account, \$18,000 in the bond fund, \$8000 in the international stock fund, and \$32,000 in the domestic stock fund.

EXAMPLE: John receives an inheritance of \$50,000. His financial advisor suggests that he invest this in three mutual funds: a money-market fund, a blue-chip stock fund, and a high-tech stock fund. The advisor estimates that the money-market fund will return 5% over the next year, the blue-chip fund 9%, and the high-tech fund 16%. John wants a total first-year return of \$4000. To avoid excessive risk, he decides to invest three times as much in the money-market fund as in the high-tech stock fund. How much should he invest in each fund?

Solution: Let

$$x = \text{amount invested in the money-market fund}$$
$$y = \text{amount invested in the blue-chip stock fund}$$
$$z = \text{amount invested in the high-tech stock fund}$$

We convert each fact given in the problem into an equation.

$$x + y + z = 50{,}000 \qquad \text{Total amount invested is \$50,000}$$
$$0.05x + 0.09y + 0.16z = 4000 \qquad \text{Total investment return is \$4000}$$
$$x = 3z \qquad \text{Money-market amount is } 3 \times \text{high-tech amount}$$

Multiplying the second equation by 100 and rewriting the third gives the following system

$$\begin{cases} x + y + z = 50,000 \\ 5x + 9y + 16z = 400,000 \\ x - 3z = 0 \end{cases}$$

Form the augmented matrix and transform it into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 50{,}000 \\ 5 & 9 & 16 & 400{,}000 \\ 1 & 0 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 50{,}000 \\ 0 & 4 & 11 & 150{,}000 \\ 0 & -1 & -4 & -50,000 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 50{,}000 \\ 0 & 4 & 11 & 150{,}000 \\ 0 & 1 & 4 & 50{,}000 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 50{,}000 \\ 0 & 1 & 4 & 50{,}000 \\ 0 & 4 & 11 & 150{,}000 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 50{,}000 \\ 0 & 1 & 4 & 50{,}000 \\ 0 & 0 & -5 & -50,000 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 50{,}000 \\ 0 & 1 & 4 & 50{,}000 \\ 0 & 0 & 1 & 10{,}000 \end{array}\right]$$

This row-echelon matrix corresponds to the system

$$\begin{cases} x + y + z = 50,000 \\ y + 4z = 50,000 \\ z = 10,000 \end{cases}$$

Use back substitution to solve this system:

$$z = 10,000$$

$$\begin{aligned} y + 4z &= 50,000 \\ y + 4(10,000) &= 50,000 \\ y + 40,000 &= 50,000 \\ y &= 10,000 \end{aligned}$$

$$\begin{aligned} x + y + z &= 50,000 \\ x + 10,000 + 10,000 &= 50,000 \\ x + 20,000 &= 50,000 \\ x &= 30,000 \end{aligned}$$

This means that John should invest

\$30,000 in the money market fund

\$10,000 in the blue-chip stock fund

\$10,000 in the high-tech stock fund

EXAMPLE: An animal feed is to be made from corn, soybeans, and cottonseed. Determine how many units of each ingredient are needed to make a feed that supplies 1800 units of fiber, 2800 units of fat, and 2200 units of protein, given that 1 unit of each ingredient provides the numbers of units shown in the table below. The table states, for example, that a unit of corn provides 10 units of fiber, 30 units of fat, and 20 units of protein.

	Corn	Soybeans	Cottonseed	Totals
Units of Fiber	10	20	30	1800
Units of Fat	30	20	40	2800
Units of Protein	20	40	25	2200

Solution: Let x represent the required number of units of corn, y the number of units of soybeans, and z the number of units of cottonseed. Since the total amount of fiber is to be 1800, we have

$$10x + 20y + 30z = 1800$$

The feed must supply 2800 units of fat, so

$$30x + 20y + 40z = 2800$$

Finally, since 2200 units of protein are required, we have

$$20x + 40y + 25z = 2200$$

Thus, we must solve this system of equations:

$$\begin{cases} 10x + 20y + 30z = 1800 \\ 30x + 20y + 40z = 2800 \\ 20x + 40y + 25z = 2200 \end{cases}$$

Form the augmented matrix and transform it into row-echelon form:

$$\left[\begin{array}{ccc|c} 10 & 20 & 30 & 1800 \\ 30 & 20 & 40 & 2800 \\ 20 & 40 & 25 & 2200 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 180 \\ 3 & 2 & 4 & 280 \\ 2 & 4 & 2.5 & 220 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 180 \\ 0 & -4 & -5 & -260 \\ 0 & 0 & -3.5 & -140 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 180 \\ 0 & 1 & 5/4 & 65 \\ 0 & 0 & 1 & 40 \end{array}\right]$$

This row-echelon matrix corresponds to the system

$$\begin{cases} x + 2y + 3z = 180 \\ y + \frac{5}{4}z = 65 \\ z = 40 \end{cases}$$

Back substitution shows that

$$z = 40$$

$$y = 65 - \frac{5}{4}(40) = 15$$

$$x = 180 - 2(15) - 3(40) = 30$$

Thus, the feed should contain 30 units of corn, 15 units of soybeans, and 40 units of cottonseed.

EXAMPLE: The table shows Census Bureau projections for the population of the United States (in millions).

Year	2020	2040	2050
U.S. Population	334	380	400

(a) Use the given data to construct a quadratic function that gives the U.S. population (in millions) in year x.

Solution: Let $x = 0$ correspond to the year 2000. Then the table represents the data points $(20, 334)$, $(40, 380)$, and $(50, 400)$. We must find a function of the form

$$f(x) = ax^2 + bx + c$$

whose graph contains these three points. If $(20, 334)$ is to be on the graph, we must have $f(20) = 334$; that is,

$$a(20)^2 + b(20) + c = 334$$
$$400a + 20b + c = 334$$

The other two points lead to these equations:

$$f(40) = 380 \qquad\qquad f(50) = 400$$
$$a(40)^2 + b(40) + c = 380 \qquad\qquad a(50)^2 + b(50) + c = 400$$
$$1600a + 40b + c = 380 \qquad\qquad 2500a + 50b + c = 400$$

Thus, we must solve this system of equations:

$$\begin{cases} 400a + 20b + c = 334 \\ 1600a + 40b + c = 380 \\ 2500a + 50b + c = 400 \end{cases} \iff \begin{cases} c + 20b + 400a = 334 \\ c + 40b + 1600a = 380 \\ c + 50b + 2500a = 400 \end{cases}$$

Form the augmented matrix and transform it into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 20 & 400 & 334 \\ 1 & 40 & 1600 & 380 \\ 1 & 50 & 2500 & 400 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 20 & 400 & 334 \\ 0 & 20 & 1200 & 46 \\ 0 & 30 & 2100 & 66 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 20 & 400 & 334 \\ 0 & 1 & 60 & 2.3 \\ 0 & 1 & 70 & 2.2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 20 & 400 & 334 \\ 0 & 1 & 60 & 2.3 \\ 0 & 0 & 10 & -.1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 20 & 400 & 334 \\ 0 & 1 & 60 & 2.3 \\ 0 & 0 & 1 & -.01 \end{array}\right]$$

This row-echelon matrix corresponds to the system

$$\begin{cases} c + 20b + 400a = 334 \\ b + 60a = 2.3 \\ a = -.01 \end{cases} \Longrightarrow \begin{array}{l} a = -.01 \\ b = 2.3 - 60(-.01) = 2.9 \\ c = 334 - 20(2.9) - 400(-.01) = 280 \end{array}$$

So the function is

$$f(x) = -.01x^2 + 2.9x + 280$$

(b) Use this model to estimate the U.S. population in the year 2030.

Solution: The year 2030 corresponds to $x = 30$, so the U.S. population is projected to be

$$f(30) = -.01(30)^2 + 2.9(30) + 280 = 358 \text{ million}$$

EXAMPLE: Kelly Karpel Kleaners sells rug-cleaning machines. The EZ model weighs 10 pounds and comes in a 10-cubic-foot box. The compact model weighs 20 pounds and comes in an 8-cubic-foot box. The commercial model weighs 60 pounds and comes in a 28-cubic-foot box. Each of Kelly's delivery vans has 248 cubic feet of space and can hold a maximum of 440 pounds. In order for a van to be fully loaded, how many of each model should it carry?

Solution: Let x be the number of EZ, y the number of compact, and z the number of commercial models carried by a van. Then we can summarize the given information in this table.

Model	**Number**	**Weight**	**Volume**
EZ	x	10	10
Compact	y	20	8
Commercial	z	60	28
Total for a load		440	248

Since a fully loaded van can carry 440 pounds and 248 cubic feet, we must solve this system of equations:

$$\begin{cases} 10x + 20y + 60z = 440 \\ 10x + 8y + 28z = 248 \end{cases}$$

Form the augmented matrix and transform it into *reduced* row-echelon form:

$$\begin{bmatrix} 10 & 20 & 60 & 440 \\ 10 & 8 & 28 & 248 \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & 20 & 60 & 440 \\ 0 & -12 & -32 & -192 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 6 & 44 \\ 0 & 1 & 8/3 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2/3 & 12 \\ 0 & 1 & 8/3 & 16 \end{bmatrix}$$

This reduced row-echelon matrix corresponds to the system

$$\begin{cases} x + \frac{2}{3}z = 12 \\ y + \frac{8}{3}z = 16 \end{cases}$$

which is easily solved:

$$\begin{cases} x = 12 - \frac{2}{3}z \\ y = 16 - \frac{8}{3}z \end{cases}$$

Hence, all solutions of the system are given by $\left(12 - \frac{2}{3}z,\ 16 - \frac{8}{3}z,\ z\right)$. The only solutions that apply in this situation, however, are those given by $z = 0, 3$, and 6, because all other values of z lead to fractions or negative numbers. (You can't deliver part of a box or a negative number of boxes). Hence, there are three ways to have a fully loaded van:

Solution	**Van Load**
(12, 16, 0)	12 EZ, 16 compact, 0 commercial
(10, 8, 3)	10 EZ, 8 compact, 3 commercial
(8, 0, 6)	8 EZ, 0 compact, 6 commercial