Section 5.3 Annuities, Future Value, and Sinking Funds

Ordinary Annuities

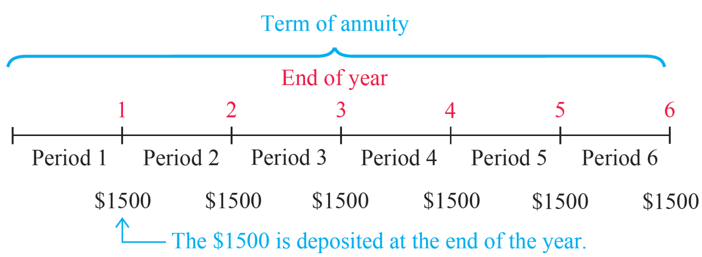
A sequence of equal payments made at equal periods of time is called an **annuity**. The time between payments is the **payment period**, and the time from the beginning of the first payment period to the end of the last period is called the **term of the annuity**. Annuities can be used to accumulate funds — for example, when you make regular deposits in a savings account. Or they can be used to pay out funds — as when you receive regular payments from a pension plan after you retire.

Annuities that pay out funds are considered in the next section. This section deals with annuities in which funds are accumulated by regular payments into an account or investment that earns compound interest. The **future value** of such an annuity is the final sum on deposit — that is, the total amount of all deposits and all interest earned by them.

We begin with **ordinary annuities** — ones where the payments are made at the end of each period and the frequency of payments is the same as the frequency of compounding the interest.

EXAMPLE: \$1500 is deposited at the end of each year for the next 6 years in an account paying 8% interest compounded annually. Find the future value of this annuity.

Solution: The Figure below shows the situation schematically.



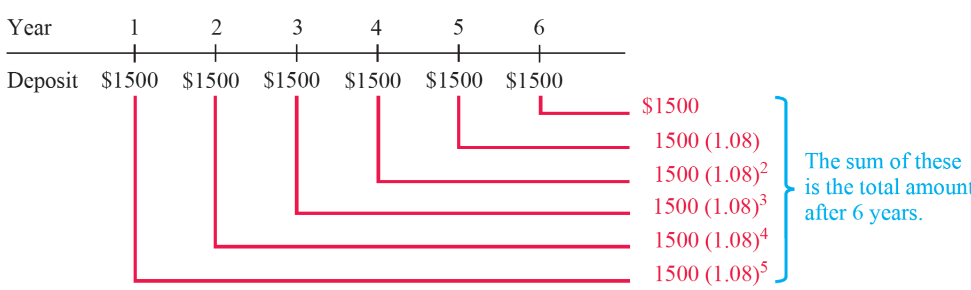
To find the future value of this annuity, look separately at each of the \$1500 payments. The first \$1500 is deposited at the end of period 1 and earns interest for the remaining 5 periods. The compound amount produced by this payment is

$$A = P(1+i)^n = 1500(1+.08)^5 = 1500(1.08)^5$$

The second \$1500 payment is deposited at the end of period 2 and earns interest for the remaining 4 periods. So the compound amount produced by the second payment is

$$1500(1+.08)^4 = 1500(1.08)^4$$

Continue to compute the compound amount for each subsequent payment, as shown in the Figure below. Note that the last payment earns no interest.



The last column of the Figure above shows that the total amount after 6 years is the sum

$$\begin{aligned} &1500 + 1500 \cdot 1.08 + 1500 \cdot 1.08^2 + 1500 \cdot 1.08^3 + 1500 \cdot 1.08^4 + 1500 \cdot 1.08^5 \\ &= 1500(1 + 1.08 + 1.08^2 + 1.08^3 + 1.08^4 + 1.08^5) \end{aligned} \tag{1}$$

It is known that

> If x is a real number and n is a positive integer, then
>
> $$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

Applying this algebraic fact to the expression in parentheses (with $x = 1.08$ and $n = 6$). It shows that the sum (the future value of the annuity) is

$$1500 \cdot \frac{1.08^6 - 1}{1.08 - 1} = \$11,003.89$$

This Example is the model for finding a formula for the future value of any annuity. Suppose that a payment of R dollars is deposited at the end of each period for n periods, at an interest rate of i per period. Then the future value of this annuity can be found by using the procedure in the Example, with these replacements:

$$\begin{array}{ccccc} 1500 & .08 & 1.08 & 6 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ R & i & 1+i & n & n-1 \end{array}$$

The future value S in the Example — call it S — is the sum (1) , which now becomes

$$S = R\left[1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-2} + (1+i)^{n-1}\right]$$

Apply the algebraic fact in the box above to the expression in brackets (with $x = 1 + i$). Then we have

$$S = R\left[\frac{(1+i)^n - 1}{(1+i) - 1}\right] = R\left[\frac{(1+i)^n - 1}{i}\right]$$

The quantity in brackets in the right-hand part of the preceding equation is sometimes written $s_{\overline{n}|i}$ (read "s-angle-n at i"). So we can summarize as follows.

> ### Future Value of an Ordinary Annuity
>
> The future value S of an ordinary annuity used to accumulate funds is given by
>
> $$\boldsymbol{S = R\left[\frac{(1+i)^n - 1}{i}\right]}, \quad \text{or} \quad \boldsymbol{S = R \cdot s_{\overline{n}|i}},$$
>
> where
>
> R is the payment at the end of each period,
>
> i is the interest rate per period, and
>
> n is the number of periods.

EXAMPLE: A rookie player in the National Football League just signed his first 7-year contract. To prepare for his future, he deposits \$150,000 at the end of each year for 7 years in an account paying 4.1% compounded annually. How much will he have on deposit after 7 years?

Solution: His payments form an ordinary annuity with $R = 150,000$, $n = 7$, and $i = .041$. The future value of this annuity is

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] = 150,000\left[\frac{(1+.041)^7 - 1}{.041}\right]$$
$$= 150,000\left[\frac{(1.041)^7 - 1}{.041}\right] = \$1,188,346.11$$

EXAMPLE: Allyson, a college professor, contributed \$950 a month to the CREF stock fund (an investment vehicle available to many college and university employees). For the past 10 years this fund has returned 4.25%, compounded monthly.

(a) How much did Allyson earn over the course of the last 10 years?

Solution: Allyson's payments form an ordinary annuity, with monthly payment $R = 950$. The interest per month is $i = \dfrac{.0425}{12}$, and the number of months in 10 years is $n = 10 \cdot 12 = 120$. The future value of this annuity is

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] = 950\left[\frac{(1+.0425/12)^{120} - 1}{.0425/12}\right] = \$141,746.90$$

(b) As of April 14, 2013, the year to date return was 9.38%, compounded monthly. If this rate were to continue, and Allyson continues to contribute \$950 a month, how much would the account be worth at the end of the next 15 years?

Solution: Deal separately with the two parts of her account (the \$950 contributions in the future and the \$141,746.90 already in the account). The contributions form an ordinary annuity as in part (a). Now we have $R = 950$, $i = .0938/12$, and $n = 12 \cdot 15 = 180$. So the future value is

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] = 950\left[\frac{(1+.0938/12)^{180} - 1}{.0938/12}\right] = \$372,068.65$$

Meanwhile, the \$141,746.90 from the first 10 years is also earning interest at 9.38%, compounded monthly. By the compound amount formula, the future value of this money is

$$A = P(1+i)^n = 141,746.90(1+.0938/12)^{180} = \$575,691.85$$

So the total amount in Allyson's account after 25 years is the sum

$$\$372,068.65 + \$575,691.85 = \$947,760.50$$

Sinking Funds

A **sinking fund** is a fund set up to receive periodic payments. Corporations and municipalities use sinking funds to repay bond issues, to retire preferred stock, to provide for replacement of fixed assets, and for other purposes. If the payments are equal and are made at the end of regular periods, they form an ordinary annuity.

EXAMPLE: A business sets up a sinking fund so that it will be able to pay off bonds it has issued when they mature. If it deposits $12,000 at the end of each quarter in an account that earns 5.2% interest, compounded quarterly, how much will be in the sinking fund after 10 years?

Solution: The sinking fund is an annuity, with $R = 12,000$, $i = .052/4$, and $n = 4(10) = 40$. The future value is

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] = 12,000\left[\frac{(1+.052/4)^{40} - 1}{.052/4}\right] = \$624,369.81$$

So there will be about $624,370 in the sinking fund.

EXAMPLE: A firm borrows $6 million to build a small factory. The bank requires it to set up a $200,000 sinking fund to replace the roof after 15 years. If the firm's deposits earn 6% interest, compounded annually, find the payment it should make at the end of each year into the sinking fund.

Solution: This situation is an annuity with future value $S = 200,000$, interest rate $i = .06$, and $n = 15$. Solve the future-value formula for R:

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] \quad \Longrightarrow \quad 200,000 = R\left[\frac{(1+.06)^{15} - 1}{.06}\right]$$

hence $R = \dfrac{200,000}{\left[\dfrac{(1+.06)^{15} - 1}{.06}\right]} = \dfrac{200,000}{23.27597} = \$8592.55$. So the annual payment is about $8593.

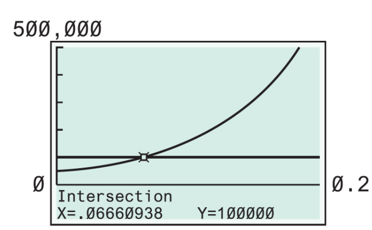
EXAMPLE: As an incentive for a valued employee to remain on the job, a company plans to offer her a $100,000 bonus, payable when she retires in 20 years. If the company deposits $200 a month in a sinking fund, what interest rate must it earn, with monthly compounding, in order to guarantee that the fund will be worth $100,000 in 20 years?

Solution: The sinking fund is an annuity with $R = 200$, $n = 12(20) = 240$, and future value $S = 100,000$. We must find the interest rate. If x is the annual interest rate in decimal form, then the interest rate per month is $i = x/12$. Inserting these values into the future-value formula, we have

$$R\left[\frac{(1+i)^n - 1}{i}\right] = S \quad \Longrightarrow \quad 200\left[\frac{(1+x/12)^{240} - 1}{x/12}\right] = 100,000$$

This equation is hard to solve algebraically. You can get a rough approximation by using a calculator and trying different values for x. With a graphing calculator, you can get an accurate solution by graphing

$$y_1 = 200\left[\frac{(1+x/12)^{240} - 1}{x/12}\right] \quad \text{and} \quad y_2 = 100,000$$

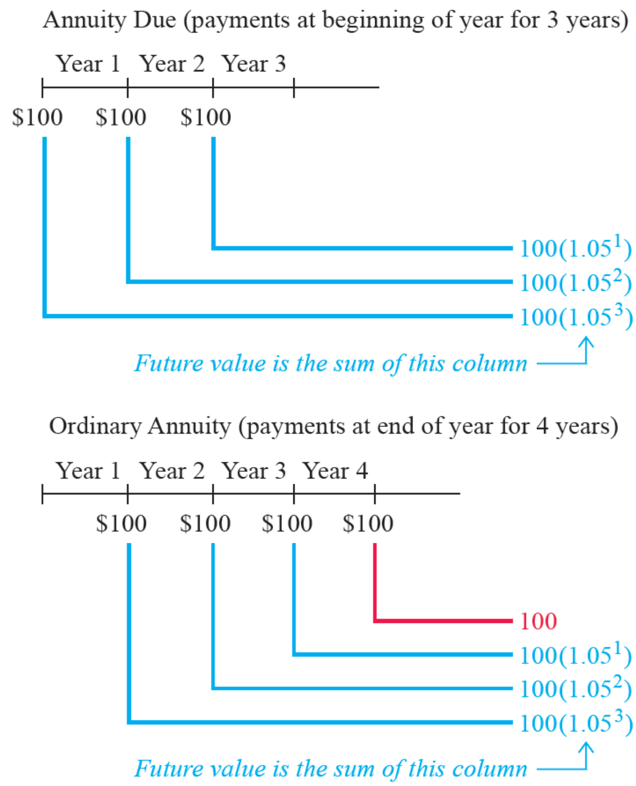


and finding the x-coordinate of the point where the graphs intersect. The Figure on the right shows that the company needs an interest rate of about 6.661%.

Annuities Due

The formula developed previously is for *ordinary annuities* — annuities with payments at the *end* of each period. The results can be modified slightly to apply to **annuities due** — annuities where payments are made at the *beginning* of each period.

An example will illustrate how this is done. Consider an annuity due in which payments of \$100 are made for 3 years, and an ordinary annuity in which payments of \$100 are made for 4 years, both with 5% interest, compounded annually. The Figure on the right computes the growth of each payment separately.



The Figure shows that the future values are the same, *except* for one \$100 payment on the ordinary annuity (shown in red). So we can use the Future Value of an Ordinary Annuity formula

$$S = R\left[\frac{(1+i)^n - 1}{i}\right]$$

to find the future value of the 4-year ordinary annuity and then subtract one \$100 payment to get the future value of the 3-year annuity due:

Future value of 3-year annuity due = **Future value of 4-year ordinary annuity** − **One payment**

$$S = 100\left[\frac{1.05^4 - 1}{.05}\right] - 100 = \$331.01.$$

Essentially the same argument works in the general case.

Future Value of an Annuity Due

The future value S of an annuity due used to accumulate funds is given by

$$S = R\left[\frac{(1+i)^{n+1} - 1}{i}\right] - R$$

S = Future value of an ordinary annuity of $n + 1$ payments − One payment,

where

R is the payment at the beginning of each period,

i is the interest rate per period, and

n is the number of periods.

EXAMPLE: Payments of \$500 are made at the beginning of each quarter for 7 years in an account paying 8% interest, compounded quarterly. Find the future value of this annuity due.

Solution: In 7 years, there are $n = 28$ quarterly periods. For an annuity due, add one period to get $n + 1 = 29$, and use the formula with $i = .08/4 = .02$:

$$S = R\left[\frac{(1+i)^{n+1}-1}{i}\right] - R = 500\left[\frac{(1+.02)^{29}-1}{.02}\right] - 500 = \$18,896.12$$

After 7 years, the account balance will be \$18,896.12.

EXAMPLE: Jay Rechtien plans to have a fixed amount from his paycheck directly deposited into an account that pays 5.5% interest, compounded monthly. If he gets paid on the first day of the month and wants to accumulate \$13,000 in the next three-and-a-half years, how much should he deposit each month?

Solution: Jay's deposits form an annuity due whose future value is $S = 13,000$. The interest rate is $i = .055/12$. There are $3 \cdot 12 + 6 = 42$ months in three-and-a-half years. Since this is an annuity due, add one period, so that $n + 1 = 43$. Then solve the future-value formula for the payment R:

$$R\left[\frac{(1+i)^{n+1}-1}{i}\right] - R = S$$

$$R\left[\frac{(1+.055/12)^{43}-1}{.055/12}\right] - R = 13,000$$

$$R\left[\frac{(1+.055/12)^{43}-1}{.055/12}\right] - R\cdot 1 = 13,000$$

$$R\left(\left[\frac{(1+.055/12)^{43}-1}{.055/12}\right] - 1\right) = 13,000$$

therefore

$$R = \frac{13,000}{\left(\left[\frac{(1+.055/12)^{43}-1}{.055/12}\right] - 1\right)} = \frac{13,000}{46.4103} = 280.110$$

Jay should have \$280.11 deposited from each paycheck.