Section 5.2 Compound Interest

With annual simple interest, you earn interest each year on your original investment. With annual **compound interest**, however, you earn interest both on your original investment and on any previously earned interest.

EXAMPLE: Suppose you deposit \$1000 at 5% annual interest. The following chart shows how your account would grow with both simple and compound interest:

End of Year	SIMPLE INTEREST Interest Earned	Balance	COMPOUND INTEREST Interest Earned	Balance
	Original Investment: \$1000		*Original Investment:* \$1000	
1	1000(.05) = \$50	\$1050	1000(.05) = \$50	\$1050
2	1000(.05) = \$50	\$1100	1050(.05) = \$52.50	\$1102.50
3	1000(.05) = \$50	\$1150	1102.50(.05) = \$55.13*	\$1157.63

As the chart shows, simple interest is computed each year on the original investment, but compound interest is computed on the entire balance at the end of the preceding year. So simple interest always produces \$50 per year in interest, whereas compound interest produces \$50 interest in the first year and increasingly larger amounts in later years (because you earn interest on your interest).

EXAMPLE: If \$7000 is deposited in an account that pays 4% interest compounded annually, how much money is in the account after nine years?

Solution: After one year, the account balance is

$$A_1 = P(1+rt) = 7000(1+.04\cdot 1) = 7000(1.04) = \$7280$$

The initial balance has grown by a factor of 1.04. At the end of the second year, the balance is

$$A_2 = A_1(1+rt) = 7280(1+.04\cdot 1) = 7280(1.04) = \$7571.20$$

The same result can be obtained in a more elegant way:

$$A_2 = A_1(1+rt) = P(1+rt)(1+rt) = P(1+rt)^2 = 7000(1+.04\cdot 1)^2 = 7000(1.04)^2 = \$7571.20$$

Similarly, at the end of the third year, the balance is

$$A_3 = A_2(1+rt) = 7571.20(1+.04\cdot 1) = 7571.20(1.04) \approx \$7874.05$$

As before, the same result can be obtained in a more elegant way:

$$A_3 = A_2(1+rt) = P(1+rt)^2(1+rt) = P(1+rt)^3 = 7000(1+.04\cdot 1)^3 = 7000(1.04)^3 \approx \$7874.05$$

This suggests that at the end of nine years, the balance is

$$7000(1.04)^9 = \$9963.18 \qquad \text{(rounded to the nearest penny)}$$

Compound Interest

If P dollars are invested at interest rate i per period, then the **compound amount** (future value) A after n compounding periods is

$$A = P(1 + i)^n.$$

REMARK: Compare this future value formula for compound interest with the one for simple interest from the previous section, using t as the number of years:

Compound interest $A = P(1 + r)^t$;

Simple interest $A = P(1 + rt)$.

The important distinction between the two formulas is that, in the compound interest formula, the number of years, t , is an *exponent*, so that money grows much more rapidly when interest is compounded.

EXAMPLE: Suppose \$1000 is deposited for twenty years in an account paying 10% per year compounded annually.

(a) Find the compound amount.

Solution: In the formula above, $P = 1000$, $i = .1$, and $n = 20$. The compound amount is

$$A = P(1+i)^n = 1000(1+0.1)^{20} = 1000(1.1)^{20} = \$6727.50$$

(b) Find the amount of interest earned.

Solution: Subtract the initial deposit from the compound amount:

$$\text{Amount of interest } = \$6727.50 - \$1000 = \$5727.50$$

	A	B	C
1	period	compound	simple
2	1	1100	1100
3	2	1210	1200
4	3	1331	1300
5	4	1464.1	1400
6	5	1610.51	1500
7	6	1771.561	1600
8	7	1948.7171	1700
9	8	2143.58881	1800
10	9	2357.947691	1900
11	10	2593.74246	2000
12	11	2853.116706	2100
13	12	3138.428377	2200
14	13	3452.271214	2300
15	14	3797.498336	2400
16	15	4177.248169	2500
17	16	4594.972986	2600
18	17	5054.470285	2700
19	18	5559.917313	2800
20	19	6115.909045	2900
21	20	6727.499949	3000

EXAMPLE: If a $16,000 investment grows to $50,000 in 18 years, what is the interest rate (assuming annual compounding)?

Solution: Use the compound interest formula, with $P = 16,000,\ A = 50,000$, and $n = 18$, and solve for i:

$$\begin{aligned} P(1+i)^n &= A \\ 16,000(1+i)^{18} &= 50,000 \\ (1+i)^{18} &= \frac{50,000}{16,000} = 3.125 \\ \sqrt[18]{(1+i)^{18}} &= \sqrt[18]{3.125} \\ 1+i &= \sqrt[18]{3.125} \\ i &= \sqrt[18]{3.125} - 1 \approx .06535 \end{aligned}$$

So the interest rate is about 6.535%.

Interest can be compounded more than once a year. Common **compounding periods** include

semiannually (2 periods per year),

quarterly (4 periods per year),

monthly (12 periods per year), and

daily (usually 365 periods per year).

When the annual interest i is compounded m times per year, the interest rate per period is understood to be i/m.

EXAMPLE: In April 2013, www.bankrate.com advertised a one-year certificate of deposit (CD) for GE Capital Retail Bank at an interest rate of 1.05%. Find the value of the CD if $10,000 is invested for one year and interest is compounded according to the given periods.

(a) Annually

Solution: Apply the formula $A = P(1+i)^n$ with $P = 10,000,\ i = .0105$, and $n = 1$:

$$A = P(1+i)^n = 10,000(1+.0105)^1 = 10,000(1.0105) = \$10,105$$

(b) Semiannually

Solution: Use the same formula and value of P. Here interest is compounded twice a year, so the number of periods is $n = 2$ and the interest rate per period is $i = \dfrac{.0105}{2}$:

$$A = P(1+i)^n = 10,000\left(1 + \frac{.0105}{2}\right)^2 = \$10,105.28$$

(c) Quarterly

Solution: Proceed as in part (b), but now interest is compounded 4 times a year, and so $n = 4$ and the interest rate per period is $i = \dfrac{.0105}{4}$:

$$A = P(1+i)^n = 10,000\left(1 + \frac{.0105}{4}\right)^4 = \$10,105.41$$

(d) Monthly

Solution: Interest is compounded 12 times a year, so $n = 12$ and $i = \dfrac{.0105}{12}$:

$$A = P(1+i)^n = 10,000\left(1 + \frac{.0105}{12}\right)^{12} = \$10,105.51$$

(e) Daily

Solution: Interest is compounded 365 times a year, so $n = 365$ and $i = \dfrac{.0105}{365}$:

$$A = P(1+i)^n = 10,000\left(1 + \frac{.0105}{365}\right)^{365} = \$10,105.55$$

EXAMPLE: The given CDs were advertised online by various banks in April 2013. Find the future value of each one. (Data from: cdrates.bankaholic.com.)

(a) Nationwide Bank: $100,000 for 5 years at 1.73% compounded daily.

Solution: Use the compound interest formula with $P = 100,000$. Interest is compounded 365 times a year, so the interest rate per period is $i = \dfrac{.0173}{365}$. Since there are five years, the number of periods in 5 years is $n = 365(5) = 1825$. The future value is

$$A = P(1+i)^n = 100,000\left(1 + \frac{.0173}{365}\right)^{1825} = \$109,034.91$$

(b) California First National Bank: $5000 for 2 years at 1.06% compounded monthly.

Solution: Use the compound interest formula with $P = 5000$. Interest is compounded 12 times a year, so the interest rate per period is $i = \dfrac{.0106}{12}$. Since there are two years, the number of periods in 2 years is $n = 12(2) = 24$. The future value is

$$A = P(1+i)^n = 5000\left(1 + \frac{.0106}{12}\right)^{24} = \$5107.08$$

The Example about GE Capital Retail Bank shows that the more often interest is compounded, the larger is the amount of interest earned. Since interest is rounded to the nearest penny, however, there is a limit on how much can be earned. In part (e) of that Example, for instance, that limit of $10,105.55 has been reached. Nevertheless, the idea of compounding more and more frequently leads to a method of computing interest called **continuous compounding** that is used in certain financial situations The formula for continuous compounding is developed in Case 5, but the formula is given in the following box where $e = 2.7182818\ldots$, which was introduced in Chapter 4 .

Continuous Compound Interest

The compound amount A for a deposit of P dollars at an interest rate r per year compounded continuously for t years is given by

$$A = Pe^{rt}.$$

EXAMPLE: Suppose that \$5000 is invested at an annual interest rate of 3.1% compounded continuously for 4 years. Find the compound amount.

Solution: In the formula for continuous compounding, let $P = 5000$, $r = .031$, and $t = 4$. Then a calculator with an e^x key shows that

$$A = Pe^{rt} = 5000e^{.031(4)} = \$5660.08$$

Ordinary corporate or municipal bonds usually make semiannual simple interest payments. With a **zero-coupon bond**, however, there are no interest payments during the life of the bond. The investor receives a single payment when the bond matures, consisting of his original investment and the interest (compounded semiannually) that it has earned. Zero-coupon bonds are sold at a substantial discount from their face value, and the buyer receives the face value of the bond when it matures. The difference between the face value and the price of the bond is the interest earned.

EXAMPLE: Doug Payne bought a 15-year zero-coupon bond paying 4.5% interest (compounded semiannually) for \$12,824.50. What is the face value of the bond?

Solution: Use the compound interest formula with $P = 12,824.50$. Interest is paid twice a year, so the rate per period is $i = .045/2$, and the number of periods in 15 years is $n = 30$. The compound amount will be the face value:

$$A = P(1+i)^n = 12,824.50(1 + .045/2)^{30} \approx \$25,000.$$

EXAMPLE: Suppose that the inflation rate is 3.5% (which means that the overall level of prices is rising 3.5% a year). How many years will it take for the overall level of prices to double?

Solution: We want to find the number of years it will take for \$1 worth of goods or services to cost \$2. Think of \$1 as the present value and \$2 as the future value, with an interest rate of 3.5%, compounded annually. Then the compound amount formula becomes

$$\begin{aligned} P(1+i)^n &= A \\ 1(1+.035)^n &= 2 \\ 1.035^n &= 2 \\ \ln 1.035^n &= \ln 2 \\ n \ln 1.035 &= \ln 2 \\ n &= \frac{\ln 2}{\ln 1.035} \approx 20.14879 \end{aligned}$$

Effective Rate (APY)

If you invest \$100 at 9%, compounded monthly, then your balance at the end of one year is

$$A = P(1+i)^n = 100\left(1+\frac{.09}{12}\right)^{12} = \$109.38$$

You have earned \$9.38 in interest, which is 9.38% of your original \$100. In other words, \$100 invested at 9.38% compounded *annually* will produce the same amount of interest (namely, $\$100 \cdot 0.0938 = \$9.38$) as does 9% compounded monthly. In this situation, 9% is called the **nominal** or **stated rate**, while 9.38% is called the **effective rate** or **annual percentage yield (APY)**.

In the discussion that follows, the nominal rate is denoted r and the APY (effective rate) is denoted r_E.

Effective Rate (r_E) or Annual Percentage Yield (APY)

The *APY* r_E is the annual compounding rate needed to produce the same amount of interest in one year, as the nominal rate does with more frequent compounding.

EXAMPLE: In April 2013, Nationwide Bank offered its customers a 5-year \$100,000 CD at 1.73% interest, compounded daily. Find the APY. (Data from: cdrates.bankaholic.com.)

Solution: The box given previously means that we must have the following:

$$\underset{\text{compounded annually}}{\$100,000 \text{ at rate } r_E} = \underset{\text{compounded daily}}{\$100,000 \text{ at } 1.73\%}$$

$$100,000(1+r_E)^1 = 100,000\left(1+\frac{.0173}{365}\right)^{365}$$

$$1+r_E = \left(1+\frac{.0173}{365}\right)^{365}$$

$$r_E = \left(1+\frac{.0173}{365}\right)^{365} - 1$$

$$r_E \approx .0175$$

So the APY is about 1.75%.

The argument in the Example above can be carried out with 100,000 replaced by P, .0173 by r, and 365 by m. The result is the effective-rate formula.

Effective Rate (APY)

The effective rate (APY) corresponding to a stated rate of interest r compounded m times per year is

$$r_E = \left(1+\frac{r}{m}\right)^m - 1.$$

EXAMPLE: When interest rates are low (as they were when this text went to press), the interest rate and the APY are insignificantly different. To see when the difference is more pronounced, we will find the APY for each of the given money market checking accounts (with balances between $50,000 and $100,000), which were advertised in October 2008 when offered rates were higher.

(a) Imperial Capital Bank: 3.35% compounded monthly.

Solution: Use the effective-rate formula with $r = .0335$ and $m = 12$:

$$\begin{aligned} r_E &= \left(1 + \frac{r}{m}\right)^m - 1 \\ &= \left(1 + \frac{.0335}{12}\right)^{12} - 1 = .034019 \end{aligned}$$

So the APY is about 3.40%, a slight increase over the nominal rate of 3.35%.

(b) U.S. Bank: 2.33% compounded daily.

Solution: Use the formula with $r = .0233$ and $m = 365$:

$$\begin{aligned} r_E &= \left(1 + \frac{r}{m}\right)^m - 1 \\ &= \left(1 + \frac{.0233}{365}\right)^{365} - 1 = .023572 \end{aligned}$$

So the APY is about 2.36%.

EXAMPLE: Bank A is now lending money at 10% interest compounded annually. The rate at Bank B is 9.6% compounded monthly, and the rate at Bank C is 9.7% compounded quarterly. If you need to borrow money, at which bank will you pay the least interest?

Solution: Compare the APYs:

$$\text{Bank } A\text{:} \quad \left(1 + \frac{.10}{1}\right)^1 - 1 = .10 = 10\%$$

$$\text{Bank } B\text{:} \quad \left(1 + \frac{.096}{12}\right)^{12} - 1 \approx .10034 = 10.034\%$$

$$\text{Bank } C\text{:} \quad \left(1 + \frac{.097}{4}\right)^4 - 1 \approx .10059 = 10.059\%$$

The lowest APY is at Bank A, which has the highest nominal rate.

Present Value for Compound Interest

The formula for compound interest, $A = P(1+i)^n$, has four variables: A, P, i, and n. Given the values of any three of these variables, the value of the fourth can be found. In particular, if A (the future amount), i, and n are known, then P can be found. Here, P is the amount that should be deposited today to produce A dollars in n periods.

EXAMPLE: Keisha Jones must pay a lump sum of $6000 in 5 years. What amount deposited today at 6.2% compounded annually will amount to $6000 in 5 years?

Solution: Here, $A = 6000$, $i = .062$, $n = 5$, and P is unknown. Substituting these values into the formula for the compound amount gives

$$6000 = P(1.062)^5 \quad \Longrightarrow \quad P = \frac{6000}{(1.062)^5} = 4441.49$$

or $4441.49. If Jones leaves $4441.49 for 5 years in an account paying 6.2% compounded annually, she will have $6000 when she needs it. To check your work, use the compound interest formula with $P = \$4441.49$, $i = .062$, and $n = 5$. You should get $A = \$6000.00$.

As the Example above shows, $6000 in 5 years is (approximately) the same as $4441.49 today (if money can be deposited at 6.2% annual interest). An amount that can be deposited today to yield a given amount in the future is called the *present value* of the future amount. By solving $A = P(1+i)^n$ for P, we get the following general formula for present value.

Present Value for Compound Interest

The **present value** of A dollars compounded at an interest rate i per period for n periods is

$$P = \frac{A}{(1+i)^n}, \quad \text{or} \quad P = A(1+i)^{-n}.$$

EXAMPLE: A zero-coupon bond with face value $15,000 and a 6% interest rate (compounded semiannually) will mature in 9 years. What is a fair price to pay for the bond today?

Solution: Think of the bond as a 9-year investment paying 6%, compounded semiannually, whose future value is $15,000. Its present value (what it is worth today) would be a fair price. So use the present value formula with $A = 15,000$. Since interest is compounded twice a year, the interest rate per period is $i = .06/2 = .03$ and the number of periods in nine years is $n = 9(2) = 18$. Hence, $P = \frac{A}{(1+i)^n} = \frac{15,000}{(1+.03)^{18}} \approx \$8810.92$.

EXAMPLE: The average annual inflation rate for the years 2010 - 2012 was 2.29%. How much did an item that sells for $1000 in early 2013 cost three years before? (Data from: inflationdata.com.)

Solution: Think of the price three years prior as the present value P and $1000 as the future value A. Then $i = .0229$, $n = 3$, and the present value is $P = \frac{A}{(1+i)^n} = \frac{1000}{(1+.0229)^3} \approx$ $934.33. So the item cost $934.33 three years prior.