Section 4.4 Logarithmic and Exponential Equations

Exponential Equations

An *exponential equation* is one in which the variable occurs in the exponent.

EXAMPLE: Solve the equation $2^x = 7$.

Solution 1: We have

$$2^x = 7$$
$$\left[\log_2 2^x = \log_2 7\right]$$
$$\left[x \log_2 2 = \log_2 7\right]$$
$$x = \log_2 7 \approx 2.807$$

Solution 2: We have

$$2^x = 7$$
$$\ln 2^x = \ln 7$$
$$x \ln 2 = \ln 7$$
$$x = \frac{\ln 7}{\ln 2} \approx 2.807$$

EXAMPLE: Solve the equation $4^{x+1} = 3$.

Solution 1: We have

$$4^{x+1} = 3$$
$$\left[\log_4 4^{x+1} = \log_4 3\right]$$
$$\left[(x+1) \log_4 4 = \log_4 3\right]$$
$$x + 1 = \log_4 3$$
$$x = \log_4 3 - 1 \approx -0.208$$

Solution 2: We have

$$4^{x+1} = 3$$
$$\ln 4^{x+1} = \ln 3$$
$$(x+1) \ln 4 = \ln 3$$
$$x + 1 = \frac{\ln 3}{\ln 4}$$
$$x = \frac{\ln 3}{\ln 4} - 1 \approx -0.208$$

EXAMPLE: Solve the equation $3^{x-3} = 5$.

EXAMPLE: Solve the equation $3^{x-3} = 5$.

Solution 1: We have

$$\begin{aligned} 3^{x-3} &= 5 \\ x - 3 &= \log_3 5 \\ x &= \log_3 5 + 3 \approx 4.465 \end{aligned}$$

Solution 2: We have

$$\begin{aligned} 3^{x-3} &= 5 \\ \ln 3^{x-3} &= \ln 5 \\ (x-3)\ln 3 &= \ln 5 \\ x - 3 &= \frac{\ln 5}{\ln 3} \\ x &= \frac{\ln 5}{\ln 3} + 3 \approx 4.465 \end{aligned}$$

EXAMPLE: Solve the equation $8e^{2x} = 20$.

Solution: We have

$$\begin{aligned} 8e^{2x} &= 20 \\ e^{2x} &= \frac{20}{8} = \frac{5}{2} \\ 2x &= \ln \frac{5}{2} \\ x &= \frac{\ln \frac{5}{2}}{2} = \frac{1}{2} \ln \frac{5}{2} \approx 0.458 \end{aligned}$$

EXAMPLE: Solve the equation $3^{2x-1} = 4^{x+2}$.

Solution: We have

$$\begin{aligned} 3^{2x-1} &= 4^{x+2} \\ \ln 3^{2x-1} &= \ln 4^{x+2} \\ (2x-1)\ln 3 &= (x+2)\ln 4 \\ 2x\ln 3 - \ln 3 &= x\ln 4 + 2\ln 4 \\ 2x\ln 3 - x\ln 4 &= \ln 3 + 2\ln 4 \\ x(2\ln 3 - \ln 4) &= \ln 3 + 2\ln 4 \\ x &= \frac{\ln 3 + 2\ln 4}{2\ln 3 - \ln 4} \approx 4.774 \end{aligned}$$

Logarithmic Equations

A *logarithmic equation* is one in which a logarithm of the variable occurs.

EXAMPLE: Solve the equation $\ln x = 8$.

Solution: We have

$$\ln x = 8$$
$$\left[e^{\ln x} = e^8\right]$$
$$x = e^8$$

EXAMPLE: Solve the equation $\log_2(x+2) = 5$.

Solution: We have

$$\log_2(x+2) = 5$$
$$\left[2^{\log_2(x+2)} = 2^5\right]$$
$$x + 2 = 2^5$$
$$x = 2^5 - 2 = 32 - 2 = 30$$

EXAMPLE: Solve the equation $\log_7(25 - x) = 3$.

Solution: We have

$$\log_7(25 - x) = 3$$
$$\left[7^{\log_7(25-x)} = 7^3\right]$$
$$25 - x = 7^3$$
$$x = 25 - 7^3 = 25 - 343 = -318$$

EXAMPLE: Solve the equation $4 + 3\log(2x) = 16$.

EXAMPLE: Solve the equation $4 + 3\log(2x) = 16$.

Solution: We have

$$\begin{aligned} 4 + 3\log(2x) &= 16 \\ 3\log(2x) &= 12 \\ \log(2x) &= 4 \\ 2x &= 10^4 \\ x &= \frac{10^4}{2} = \frac{10,000}{2} = 5,000 \end{aligned}$$

EXAMPLE: Solve the equation $\log(x + 2) + \log(x - 1) = 1$.

Solution: We have

$$\begin{aligned} \log(x + 2) + \log(x - 1) &= 1 \\ \log[(x + 2)(x - 1)] &= 1 \\ (x + 2)(x - 1) &= 10 \\ x^2 + x - 2 &= 10 \\ x^2 + x - 12 &= 0 \\ (x + 4)(x - 3) &= 0 \end{aligned}$$

$$\begin{aligned} x + 4 &= 0 \qquad\qquad \text{or} \qquad\qquad & x - 3 &= 0 \\ x &= -4 & x &= 3 \end{aligned}$$

We check these potential solutions in the original equation and find that $x = -4$ is not a solution (because logarithms of negative numbers are undefined), but $x = 3$ is a solution.

REMARK: For more examples, see the Appendix.

Applications

EXAMPLE: A sum of \$5000 is invested at an interest rate of 5% per year. Find the time required for the money to double if the interest is compounded according to the following method.

(a) Semiannual (b) Continuous

Solution:
(a) We use the formula for compound interest (will be discussed in Chapter 5)

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

with $P = \$5000,\ A(t) = \$10,000,\ r = 0.05,\ n = 2$ and solve the resulting exponential equation for t.

$$5000\left(1 + \frac{0.05}{2}\right)^{2t} = 10,000$$

$$(1.025)^{2t} = 2$$

$$\log 1.025^{2t} = \log 2$$

$$2t \log 1.025 = \log 2$$

$$t = \frac{\log 2}{2 \log 1.025} \approx 14.04$$

The money will double in 14.04 years.

(b) We use the formula for continuously compounded interest (will be discussed in Chapter 5)

$$A(t) = Pe^{rt}$$

with $P = \$5000,\ A(t) = \$10,000,\ r = 0.05$ and solve the resulting exponential equation for t.

$$5000e^{0.05t} = 10,000$$

$$e^{0.05t} = 2$$

$$0.05t = \ln 2$$

$$t = \frac{\ln 2}{0.05} \approx 13.86$$

The money will double in 13.86 years.

EXAMPLE: The number of total subscribers (in millions) to Netflix, Inc. can be approximated by the function

$$g(x) = 9.78(1.07)^x$$

where $x = 1$ corresponds to the first quarter of the year 2009, $x = 2$ corresponds to the second quarter of the year 2009, etc. Assume the model remains accurate and determine when the number of subscribers reached 28 million. (Data from: *The Wall Street Journal and The Associated Press.*)

Solution: You are being asked to find the value of x for which $g(x) = 28$ — that is, to solve the following equation:

$$9.78(1.07)^x = 28$$

$$(1.07)^x = \frac{28}{9.78}$$

$$\ln(1.07)^x = \ln\left(\frac{28}{9.78}\right)$$

$$x\ln(1.07) = \ln\left(\frac{28}{9.78}\right)$$

$$x = \frac{\ln\left(\frac{28}{9.78}\right)}{\ln(1.07)} \approx 15.5$$

The 15th quarter corresponds to the third quarter of the year 2012. Hence, according to this model, Netflix, Inc., reached 28 million subscribers in the third quarter of 2012.

EXAMPLE: One action that government could take to reduce carbon emissions into the atmosphere is to place a tax on fossil fuels. This tax would be based on the amount of carbon dioxide that is emitted into the air when such a fuel is burned. The *cost-benefit* equation

$$\ln(1 - P) = -.0034 - .0053T$$

describes the approximate relationship between a tax of T dollars per ton of carbon dioxide and the corresponding percent reduction P (in decimals) in em1ssions of carbon dioxide.

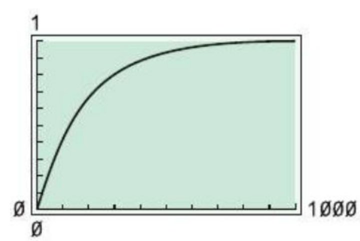
(a) Write P as a function of T.

Solution We begin by writing the cost-benefit equation in exponential form:

$$\ln(1 - P) = -.0034 - .0053T$$

$$1 - P = e^{-.0034 - .0053T}$$

$$P = 1 - e^{-.0034 - .0053T}$$



A calculator-generated graph of $P(T)$ is shown in the Figure on the right.

(b) Discuss the benefit of continuing to raise taxes on carbon dioxide emissions.

Solution From the graph, we see that initially there is a rapid reduction in carbon dioxide emissions. However, after a while, there is little benefit in raising taxes further.

Appendix

EXAMPLE: Solve the following equations

(a) $\log(x+8)+\log(x-1)=1$ (b) $\log(x^2-1)-\log(x+1)=3$

Solution:

(a) We have

$$\log(x+8)+\log(x-1)=1$$

$$\log[(x+8)(x-1)]=1$$

$$(x+8)(x-1)=10$$

$$x^2+7x-8=10$$

$$x^2+7x-18=0$$

$$(x+9)(x-2)=0$$

$$x+9=0 \qquad \text{or} \qquad x-2=0$$

$$x=-9 \qquad\qquad x=2$$

We check these potential solutions in the original equation and find that $x=-9$ is not a solution (because logarithms of negative numbers are undefined), but $x=2$ is a solution.

(b) We have

$$\log(x^2-1)-\log(x+1)=3$$

$$\log\frac{x^2-1}{x+1}=3$$

$$\frac{x^2-1}{x+1}=10^3$$

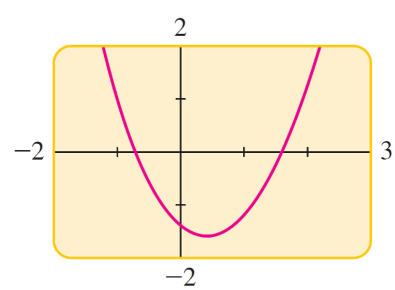
$$\frac{(x-1)(x+1)}{x+1}=1000$$

$$x-1=1000$$

$$x=1001$$

EXAMPLE: Solve the equation $x^2=2\ln(x+2)$ graphically.

Solution: We first move all terms to one side of the equation $x^2-2\ln(x+2)=0$. Then we graph $y=x^2-2\ln(x+2)$. The solutions are the x-intercepts of the graph.



EXAMPLE: Find the solution of the equation, correct to two decimal places.

(a) $10^{x+3} = 6^{2x}$ (b) $5\ln(3-x) = 4$ (c) $\log_2(x+2) + \log_2(x-1) = 2$

Solution:

(a) We have

$$\begin{aligned} 10^{x+3} &= 6^{2x} \\ \ln 10^{x+3} &= \ln 6^{2x} \\ (x+3)\ln 10 &= 2x\ln 6 \\ x\ln 10 + 3\ln 10 &= 2x\ln 6 \\ x\ln 10 - 2x\ln 6 &= -3\ln 10 \\ x(\ln 10 - 2\ln 6) &= -3\ln 10 \\ x &= \frac{-3\ln 10}{\ln 10 - 2\ln 6} \approx 5.39 \end{aligned}$$

(b) We have

$$\begin{aligned} 5\ln(3-x) &= 4 \\ \ln(3-x) &= \frac{4}{5} \\ 3-x &= e^{4/5} \\ x &= 3 - e^{4/5} \approx 0.77 \end{aligned}$$

(c) We have

$$\begin{aligned} \log_2(x+2) + \log_2(x-1) &= 2 \\ \log_2(x+2)(x-1) &= 2 \\ (x+2)(x-1) &= 4 \\ x^2 + x - 2 &= 4 \\ x^2 + x - 6 &= 0 \\ (x-2)(x+3) &= 0 \end{aligned}$$

$$x - 2 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 2 \qquad\qquad x = -3$$

Since $x = -3$ is not from the domain of $\log_2(x+2) + \log_2(x-1)$, the only answer is $x = 2$.