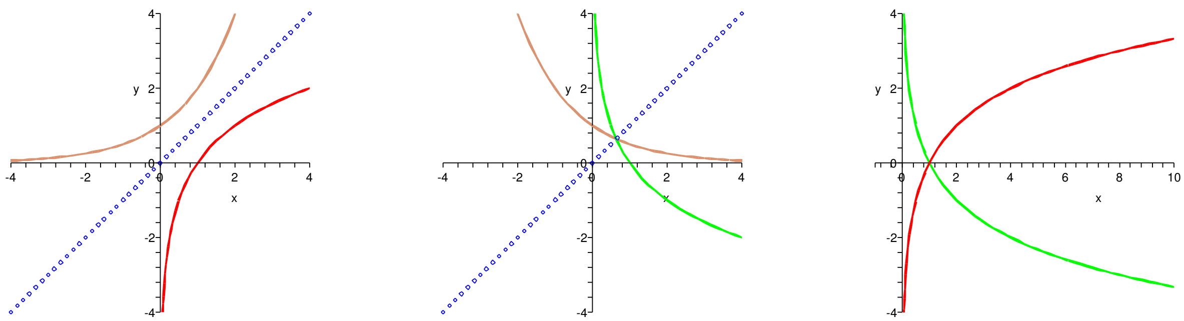
Section 4.3 Logarithmic Functions

DEFINITION: Let a be a positive number with $a \neq 1$. The **logarithmic function with base** $\boldsymbol{a}$, denoted by $\mathbf{log}_{\boldsymbol{a}}$, is defined by

$$\boxed{\log_a x = y \quad \Longleftrightarrow \quad a^y = x}$$

So, $\log_a x$ is the *exponent* to which the base a must be raised to give x.



BASIC PROPERTIES: $f(x) = \log_a x$ is a continuous function with domain $(0, \infty)$ and range $(-\infty, \infty)$. Moreover,

$$\boxed{\log_a(a^x) = x \text{ for every } x \in \mathbb{R}, \quad a^{\log_a x} = x \text{ for every } x > 0}$$

REMARK: It immediately follows from property 1 that

$$\boxed{\log_a a = 1, \quad \log_a 1 = 0}$$

EXAMPLES:

1. $\log_2 2 = 1, \ \log_2 4 = 2, \ \log_2 8 = 3, \ \log_2 16 = 4$

2. $\log_3 3 = 1, \ \log_3 9 = 2, \ \log_3 27 = 3, \ \log_3 81 = 4$

3. $\log_3\left(\dfrac{1}{3}\right) = \log_3 3^{-1} = -1, \ \log_3\left(\dfrac{1}{9}\right) = \log_3 3^{-2} = -2, \ \log_3\left(\dfrac{1}{27}\right) = \log_3 3^{-3} = -3$

4. $\log_5 \sqrt{5} = \log_5 5^{1/2} = \dfrac{1}{2}, \ \log_7 \sqrt[3]{7} = \log_7 7^{1/3} = \dfrac{1}{3}$

5. $\log_{11}\left(\dfrac{1}{\sqrt[5]{11}}\right) = \log_{11}\left(\dfrac{1}{11^{1/5}}\right) = \log_{11} 11^{-1/5} = -\dfrac{1}{5}$

6. $\log_4 8 = \log_4 2^3 = \log_4(4^{1/2})^3 = \log_4 4^{(1/2)\cdot 3} = \log_4 4^{3/2} = \dfrac{3}{2}$

7. $\log_2 3 \approx 1.58496, \ \log_3 5 \approx 1.46497, \ \log_7 1000 \approx 3.54988$

Natural and Common Logarithms

DEFINITION: The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\boxed{\log_e x = \ln x}$$

BASIC PROPERTIES:

1. $\ln(e^x) = x$ for every $x \in \mathbb{R}$.
2. $e^{\ln x} = x$ for every $x > 0$.

REMARK: It immediately follows from property 1 that

$$\ln e = 1$$

EXAMPLES: $\ln e^2 = 2, \quad \ln \sqrt[3]{e} = \ln(e^{1/3}) = \dfrac{1}{3}, \quad \ln(1/e) = \ln(e^{-1}) = -1, \quad \ln 3 \approx 1.09861$

DEFINITION: The logarithm with base 10 is called the **common logarithm** and has a special notation:

$$\boxed{\log_{10} x = \log x}$$

BASIC PROPERTIES:

1. $\log(10^x) = x$ for every $x \in \mathbb{R}$.
2. $10^{\log x} = x$ for every $x > 0$.

REMARK: It immediately follows from property 1 that

$$\log 10 = 1$$

EXAMPLES:

1. $\log 100 = \log 10^2 = 2, \quad \log 1,000,000,000 = \log 10^9 = 9, \quad \log 10^{100} = 100$
2. $\log 0.01 = \log 10^{-2} = -2, \quad \log \sqrt{10} = \log 10^{1/2} = \dfrac{1}{2}, \quad \log \dfrac{1}{\sqrt[7]{10}} = \log \dfrac{1}{10^{1/7}} = \log 10^{-1/7} = -\dfrac{1}{7}$
3. $\log 3 \approx 0.47712, \ \log 15 \approx 1.17609, \ \log 101 \approx 2.00432, \ \log 0.2 \approx -0.69897$

Properties of Logarithms

LAWS OF LOGARITHMS: If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r\log_a x$ where r is any real number.

EXAMPLES:

1. Use the laws of logarithms to evaluate $\log_2 4$.

Solution: We have

$$\log_2 4 = \log_2 2^2 = 2\log_2 2 = 2\cdot 1 = 2$$

2. Use the laws of logarithms to evaluate $\log 1,000,000$.

Solution: We have

$$\log 1,000,000 = \log 10^6 = 6\log 10 = 6\cdot 1 = 6$$

3. Use the laws of logarithms to evaluate $\log_7\sqrt{7}$.

Solution: We have

$$\log_7\sqrt{7} = \log_7 7^{1/2} = \frac{1}{2}\log_7 7 = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

4. Use the laws of logarithms to evaluate $\log_5 \dfrac{1}{\sqrt[3]{5}}$.

Solution: We have

$$\log_5\frac{1}{\sqrt[3]{5}} = \log_5\frac{1}{5^{1/3}} = \log_5 5^{-1/3} = -\frac{1}{3}\log_5 5 = -\frac{1}{3}\cdot 1 = -\frac{1}{3}$$

or

$$\log_5\frac{1}{\sqrt[3]{5}} = \log_5\frac{1}{5^{1/3}} = \log_5 1 - \log_5 5^{1/3} = 0 - \frac{1}{3}\log_5 5 = -\frac{1}{3}\log_5 5 = -\frac{1}{3}\cdot 1 = -\frac{1}{3}$$

5. Use the laws of logarithms to evaluate $\log_4 8$.

Solution: We have

$$\log_4 8 = \log_4 2^3 = \log_4(4^{1/2})^3 = \log_4 4^{(1/2)\cdot 3} = \log_4 4^{3/2} = \frac{3}{2}\log_4 4 = \frac{3}{2}\cdot 1 = \frac{3}{2}$$

6. Use the laws of logarithms to evaluate $\log_3 270 - \log_3 10$.

Solution: We have

$$\log_3 270 - \log_3 10 = \log_3\left(\frac{270}{10}\right) = \log_3 27 = \log_3 3^3 = 3\log_3 3 = 3\cdot 1 = 3$$

7. Use the laws of logarithms to evaluate $\log_2 12 + \log_2 3 - \log_2 9$.

7. Use the laws of logarithms to evaluate $\log_2 12 + \log_2 3 - \log_2 9$.

Solution: We have

$$\log_2 12 + \log_2 3 - \log_2 9 = \log_2(12 \cdot 3) - \log_2 9 = \log_2\left(\frac{12 \cdot 3}{9}\right) = \log_2 4 = \log_2 2^2$$

$$= 2\log_2 2 = 2 \cdot 1 = 2$$

EXAMPLES:

1. $\ln\left(x(x+1)\right) = \ln x + \ln(x+1)$

2. $\ln\left(\dfrac{x}{x+2}\right) = \ln x - \ln(x+2)$

3. $$\ln\left(\frac{x(x+1)}{(x+2)(x-5)}\right) = \ln\Big(x(x+1)\Big) - \ln\Big((x+2)(x-5)\Big)$$
$$= \Big(\ln x + \ln(x+1)\Big) - \Big(\ln(x+2) + \ln(x-5)\Big)$$
$$= \ln x + \ln(x+1) - \ln(x+2) - \ln(x-5)$$

4. $$\ln\sqrt[3]{7a^3b^5} = \ln(7a^3b^5)^{1/3} = \frac{1}{3}\ln(7a^3b^5) = \frac{1}{3}(\ln 7 + \ln a^3 + \ln b^5) = \frac{1}{3}(\ln 7 + 3\ln a + 5\ln b)$$
$$= \frac{1}{3}\ln 7 + \ln a + \frac{5}{3}\ln b$$

5. $$\ln\left(\frac{x^2}{\sqrt[5]{y^3z}}\right) = \ln\left(\frac{x^2}{(y^3z)^{1/5}}\right) = \ln x^2 - \ln\left(y^3z\right)^{1/5} = 2\ln x - \frac{1}{5}\ln\left(y^3z\right)$$
$$= 2\ln x - \frac{1}{5}\left(\ln y^3 + \ln z\right)$$
$$= 2\ln x - \frac{1}{5}(3\ln y + \ln z)$$
$$= 2\ln x - \frac{3}{5}\ln y - \frac{1}{5}\ln z$$

Change of Base

IMPORTANT FORMULA: For any positive a and b $(a, b \neq 1)$ we have

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

In particular, if $a = e$ or 10, then

$$\boxed{\log_b x = \frac{\ln x}{\ln b} = \frac{\log x}{\log b}}$$

EXAMPLES: $\log_5 4 = \dfrac{\ln 4}{\ln 5} = \dfrac{\log 4}{\log 5} \approx 0.86135$, $\quad \log_4 8 = \log_{2^2} 2^3 = \dfrac{\log 2^3}{\log 2^2} = \dfrac{3\log 2}{2\log 2} = \dfrac{3}{2}$

Applications

EXAMPLE: The life expectancy at birth of a person born in year x is approximated by the function

$$f(x) = 17.6 + 12.8 \ln x$$

where $x = 10$ corresponds to 1910. (Data from: U.S. National Center for Health Statistics.)

(a) Find the life expectancy of persons born in 1910, 1960, and 2010.

Solution: Since these years correspond to $x = 10, x = 60$, and $x = 110$, respectively, use a calculator to evaluate $f(x)$ at these numbers:

$$f(10) = 17.6 + 12.8 \ln(10) = 47.073$$

$$f(60) = 17.6 + 12.8 \ln(60) = 70.008$$

$$f(110) = 17.6 + 12.8 \ln(110) = 77.766$$

So in the half-century from 1910 to 1960, life expectancy at birth increased from about 47 to 70 years, an increase of 23 years. But in the half-century from 1960 to 2010, it increased less than 8 years, from about 70 to 77.8 years.

(b) If this function remains accurate, when will life expectancy at birth be 80.2 years?

Solution We must solve the equation $f(x) = 80.2$, that is,

$$17.6 + 12.8 \ln x = 80.2$$

In the next section, we shall see how to do this algebraically. For now, we solve the equation graphically by graphing $f(x)$ and $y = 80.2$ on the same screen and finding their intersection point. The Figure below shows that the x-coordinate of this point is approximately 133. So life expectancy will be 80.2 years in 2033.