Section 4.2 Applications of Exponential Functions

In many situations in biology, economics, and the social sciences, a quantity changes at a rate proportional to the quantity present. For example, a country's population might be increasing at a rate of 1.3% a year. Tn such cases, the amount present at time t is given by an **exponential growth function**.

It is understood that growth can involve either growing larger or growing smaller.

Exponential Growth Function

Under normal conditions, growth can be described by a function of the form

$$f(t) = y_0 e^{kt} \quad \text{or} \quad f(t) = y_0 b^t,$$

where $f(t)$ is the amount present at time t, y_0 is the amount present at time $t = 0$, and k and b are constants that depend on the rate of growth.

When $f(t) = y_0 e^{kt}$, and $k > 0$, we describe $f(t)$ as modeling exponential growth, and when $k < 0$, we describe $f(t)$ as modeling exponential decay. When $f(t) = y_0 b^t$, and $b > 1$, we describe $f(t)$ as modeling exponential growth, and when $0 < b < 1$, we describe $f(t)$ as modeling exponential decay.

EXAMPLE: Since the early 1970s, the amount of the total credit market (debt owed by the government, companies, or individuals) as a percentage of gross domestic product (GDP) can be approximated by the exponential function

$$f(t) = y_0 e^{.02t}$$

where t is time in years, $t = 0$ corresponds to the year 1970, and $f(t)$ is a percent. (Data from: Federal Reserve.)

(a) If the amount of total credit market was 155% of the GDP in 1970, find the percent in the year 2005.

Solution: Since y_0 represents the percent when $t = 0$ (that is, in 1970) we have $y_0 = 155$. So the growth function is $f(t) = 155e^{.02t}$. To find the percent of the total credit market in the year 2005, evaluate $f(t)$ at $t = 35$ (which corresponds to the year 2005):

$$f(t) = 155e^{.02t}$$

$$f(35) = 155e^{.02(35)} \approx 312$$

Hence, the percent of the total credit market in the year 2005 was approximately 312% of GDP.

(b) If the model remains accurate, what percent of GDP will the total credit market be in the year 2015?

Solution: Since 2015 corresponds to $t = 45$, evaluate the function at $t = 45$:

$$f(45) = 155e^{.02(45)} \approx 381\%$$

EXAMPLE: Cigarette consumption in the United States has been decreasing for some time. Based on data from the Centers for Disease Control and Prevention, the number (in billions) of cigarettes consumed can be approximated by the function

$$g(x) = 436e^{-.036x}$$

with $x = 0$ corresponding to the year 2000.

(a) Find the number of cigarettes consumed in the years 2005 and 2010.

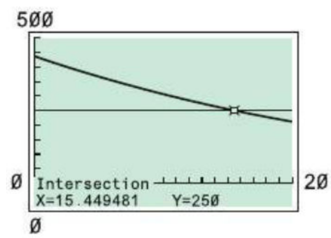
Solution: The years 2005 and 2010 correspond to $x = 5$ and $x = 10$, respectively. We can evaluate $g(x)$ at these values for x:

$$g(5) = 436e^{-.036(5)} \approx 364 \text{ billion in the year 2005}$$

$$g(10) = 436e^{-.036(10)} \approx 304 \text{ billion in the year 2010}$$

(b) If this model remains accurate, when will cigarette consumption fall to 250 billion?

Solution: Graph $y = 436e^{-.036x}$, and $y = 250$ on the same screen and find the x-coordinate of their intersection point. The Figure below shows that consumption is expected to be 250 billion in the year 2015 ($x = 15$).



EXAMPLE: When money is placed in a bank account that pays compound interest, the amount in the account grows exponentially, as we shall see in Chapter 5. Suppose such an account grows from \$1000 to \$1316 in 7 years.

(a) Find a growth function of the form $f(t) = y_0 b^t$ that gives the amount in the account at time t years.

Solution: The values of the account at time $t = 0$ and $t = 7$ are given; that is, $f(0) = 1000$ and $f(7) = 1316$. Solve the first of these equations for y_0:

$$f(0) = 1000 \quad \Longrightarrow \quad y_0 b^0 = 1000 \quad \Longrightarrow \quad y_0 = 1000$$

So the rule of f has the form $f(t) = 1000b^t$. Now solve the equation $f(7) = 1316$ for b:

$$\begin{aligned} f(7) &= 1316 \\ 1000b^7 &= 1316 \\ b^7 &= \frac{1316}{1000} = 1.316 \\ \sqrt[7]{b^7} &= \sqrt[7]{1.316} \\ b &= (1.316)^{1/7} \approx 1.04 \end{aligned}$$

So the rule of the function is $f(t) = 1000(1.04)^t$.

(b) How much is in the account after 12 years?

Solution: $f(12) = 1000(1.04)^{12} = \$1601.03$.

Other Exponential Models

When a quantity changes exponentially, but does not either grow very large or decrease practically to 0, as in the previous Examples, different functions are needed.

EXAMPLE: Sales of a new product often grow rapidly at first and then begin to level off with time. Suppose the annual sales of an inexpensive can opener are given by

$$S(x) = 10,000(1 - e^{-.5t})$$

where $x = 0$ corresponds to the time the can opener went on the market.

(a) What were the sales in each of the first three years?

Solution: At the end of one year ($x = 1$), sales were

$$S(1) = 10,000(1 - e^{-.5(1)}) \approx 3935$$

Sales in the next two years were

$$S(2) = 10,000(1 - e^{-.5(2)}) \approx 6321 \quad \text{and} \quad S(3) = 10,000(1 - e^{-.5(3)}) \approx 7769$$

(b) What were the sales at the end of the 10th year?

Solution: $S(10) = 10,000(1 - e^{-.5(10)}) \approx 9933.$

(c) Graph the function S. What does it suggest?

Solution: The graph can be obtained by plotting points and connecting them with a smooth curve or by using a graphing calculator, as in the Figure below. The graph indicates that sales will level off after the 12th year, to around 10,000 can openers per year.