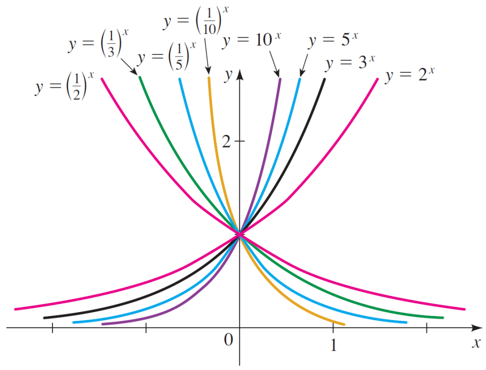
Section 4.1 Exponential Functions

DEFINITION: An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant.

x	$\left(\frac{1}{10}\right)^x$
-3	$\left(\frac{1}{10}\right)^{-3} = 10^3 = 1000$
-2	$\left(\frac{1}{10}\right)^{-2} = 10^2 = 100$
-1	$\left(\frac{1}{10}\right)^{-1} = 10^1 = 10$
0	$\left(\frac{1}{10}\right)^{0} = 1$
1	$\left(\frac{1}{10}\right)^{1} = \frac{1}{10} = 0.1$
2	$\left(\frac{1}{10}\right)^{2} = \frac{1}{10^2} = 0.01$
3	$\left(\frac{1}{10}\right)^{3} = \frac{1}{10^3} = 0.001$



x	10^x
-3	$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$
-2	$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$
-1	$10^{-1} = \frac{1}{10} = 0.1$
0	$10^0 = 1$
1	$10^1 = 10$
2	$10^2 = 100$
3	$10^3 = 1000$

REMARKS:

1. The graph is above them x-axis.
2. The y-intercept is 1.
3. The graph climbs steeply to the right $(a > 1)$ or left $(0 < a < 1)$.
4. The larger the base a, the more steeply the graph rises to the right $(a > 1)$ or left $(0 < a < 1)$.
5. The x-axis is a horizontal asymptote of $f(x) = a^x$.

BASIC ALGEBRAIC PROPERTIES:

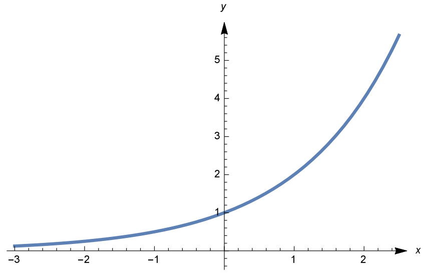
1. $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$ if n is a positive integer.
2. $a^0 = 1$.
3. $a^{-n} = \dfrac{1}{a^n}$.
4. $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

THEOREM: If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $a^x > 0$ for all x. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \dfrac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

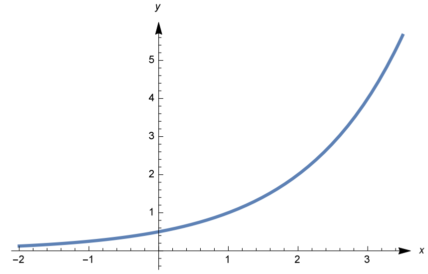
EXAMPLE: Graph the following functions:

(a) $f(x) = 2^{x-1}$



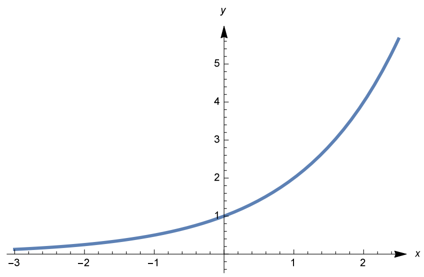
$y = 2^x$

$\Longrightarrow$



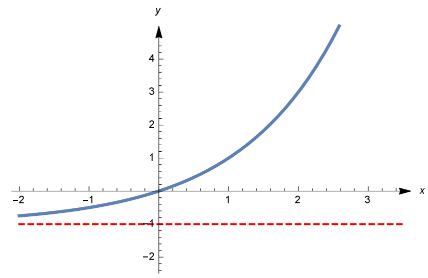
$f(x) = 2^{x-1}$ (horizontal shift)

(b) $g(x) = 2^x - 1$



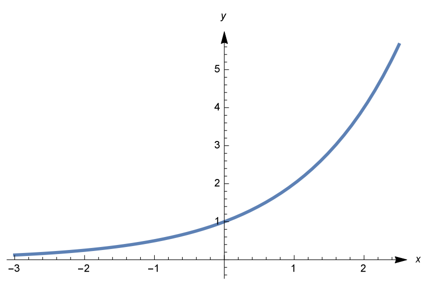
$y = 2^x$

$\Longrightarrow$



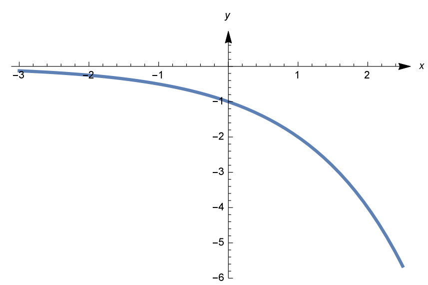
$g(x) = 2^x - 1$ (vertical shift)

(c) $h(x) = -2^x$



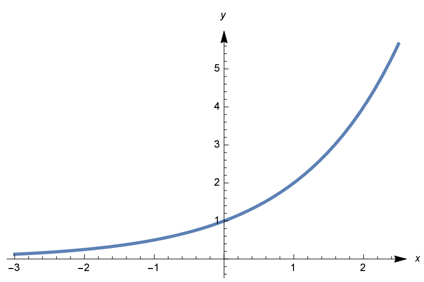
$y = 2^x$

$\Longrightarrow$



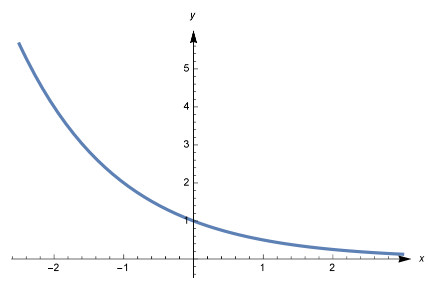
$h(x) = -2^x$ (reflection)

(d) $p(x) = 2^{-x}$



$y = 2^x$

$\Longrightarrow$



$p(x) = 2^{-x}$ (reflection)

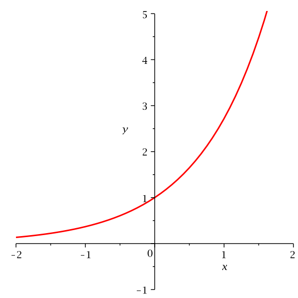
REMARK: An other way to graph $p(x)$ is to rewrite it as $p(x) = \dfrac{1}{2^x} = \left(\dfrac{1}{2}\right)^x$, which gives the same result by the Figure on page 1.

The Number e

It is known that

$$e \approx 2.718281828459045235360287471352662497757247093 6...$$

x	$\left(1+\dfrac{1}{x}\right)^x$	Value
1	2^1	2
10	1.1^{10}	2.593742460
100	1.01^{100}	2.704813829
1000	1.001^{1000}	2.716923932
10000	1.0001^{10000}	2.718145927

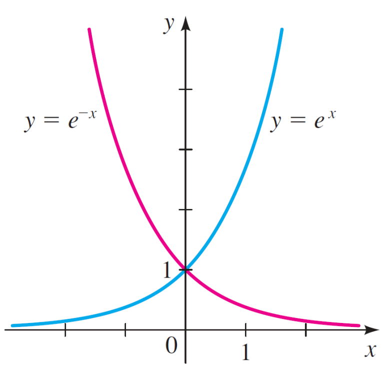


DEFINITION: The **natural exponential function** is $f(x) = e^x$.

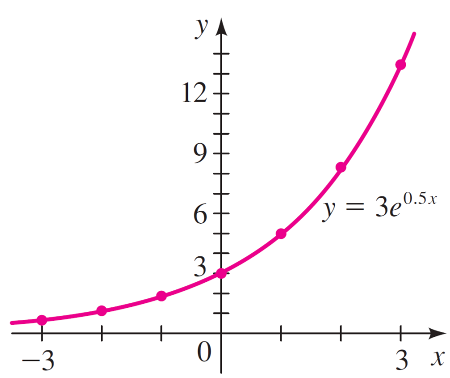
EXAMPLE: Sketch the graph of each function.
(a) $f(x) = e^{-x}$ (b) $g(x) = 3e^{0.5x}$

Solution:
(a) We start with the graph of $y = e^x$ and reflect in the y-axis to obtain the graph of $y = e^{-x}$.

(b) We calculate several values, plot the resulting points, then connect the points with a smooth curve.



x	$f(x) = 3e^{0.5x}$
−3	0.67
−2	1.10
−1	1.82
0	3.00
1	4.95
2	8.15
3	13.45



Applications

EXAMPLE: The amount of wine (in millions of gallons) consumed in the United States can be approximated by the function $f(x) = 139.6e^{.031x}$, where $x = 0$ corresponds to the year 1950. (Data from: www.wineinstitute.org.)

(a) How much wine was consumed in the year 1970?

Solution: Since 1970 corresponds to $x = 20$, we evaluate $f(20)$:

$$f(20) = 139.6e^{.031(20)} = 260 \text{ million gallons}$$

So the consumption was approximately 260 million gallons.

(b) How much wine was consumed in the year 2010?

Solution: Since 2010 corresponds to $x = 60$, we evaluate $f(60)$:

$$f(60) = 139.6e^{.031(60)} = 897 \text{ million gallons}$$

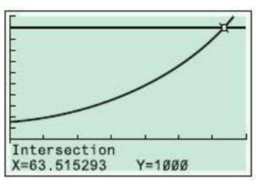
So the consumption was approximately 897 million gallons.

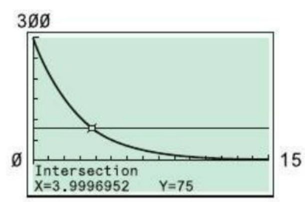
(c) Use a graphing calculator to determine when consumption will reach 1000 million gallons.

Solution: Since f measures consumption in millions of gallons, we must solve the equation $f(x) = 1000$, that is

$$139.6e^{.031x} = 1000$$

One way to do this is to find the intersection point of the graphs of $y = 139.6e^{.031x}$ and $y = 1000$. (The calculator's intersection finder is in the same menu as its root [or zero] finder.) The Figure below (left) shows that this point is approximately $(63.5, 1000)$. Hence, consumption reached 1000 million gallons when $x = 63.5$, that is, in 2013.





EXAMPLE: When a patient is given a 300-mg dose of the drug cimetidine intravenously, the amount C of the drug in the bloodstream t hours later is given by $C(t) = 300e^{-.3466t}$.

(a) How much of the drug is in the bloodstream after 3 hours and after 10 hours?

Solution: Evaluate the function at $t = 3$ and $t = 10$:

$$C(3) = 300e^{-.3466(3)} = 106.1 \text{ mg}$$

$$C(10) = 300e^{-.3466(10)} = 9.4 \text{ mg}$$

(b) Doctors want to give a patient a second 300-mg dose of cimetidine when its level in her bloodstream decreases to 75 mg. Use graphing technology to determine when this should be done.

Solution: The second dose should be given when $C(t) = 75$, so we must solve the equation

$$300e^{-.3466t} = 75$$

To this end, we graph $y = 300e^{-.3466t}$ and $y = 75$ and find their intersection point. The Figure above (right) shows that this point is approximately $(4, 75)$. So the second dose should be given 4 hours after the first dose.

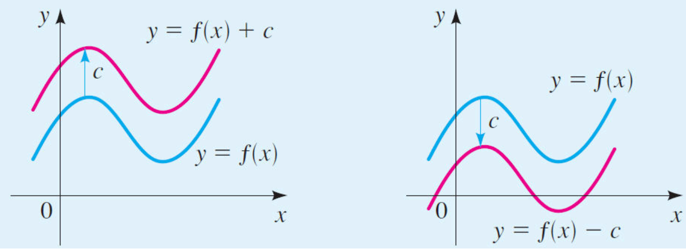
Appendix (Transformations of Functions)

Vertical Shifts of Graphs

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

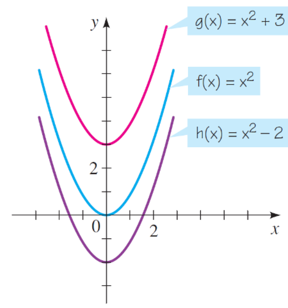
To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



EXAMPLE: Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = x^2 + 3$

(b) $h(x) = x^2 - 2$

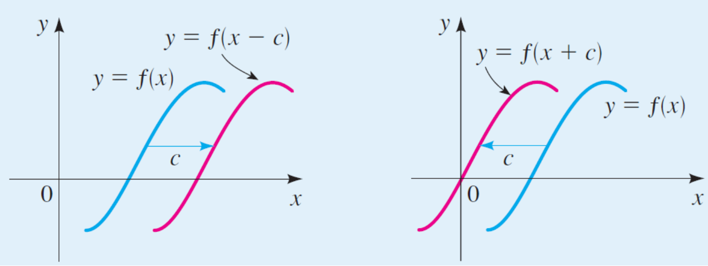


Horizontal Shifts of Graphs

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

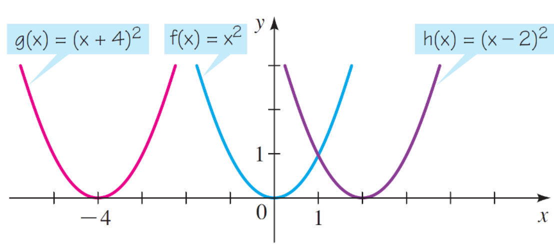
To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.



EXAMPLE: Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = (x + 4)^2$

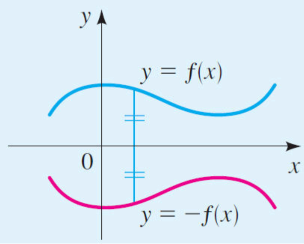
(b) $h(x) = (x - 2)^2$

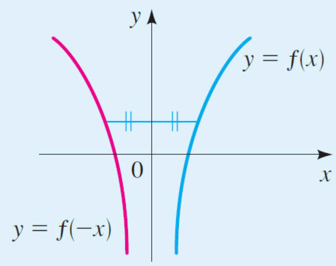


Reflecting Graphs

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

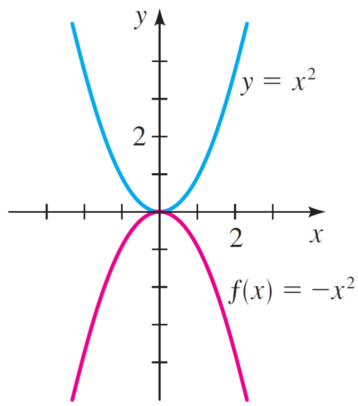
To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.



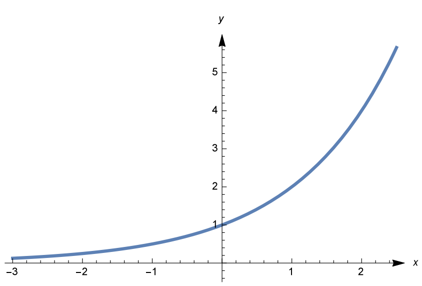


EXAMPLE: Sketch the graph of each function.

(a) $f(x) = -x^2$

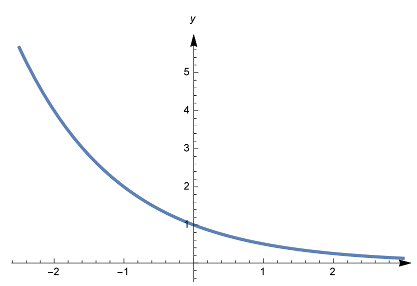


(b) $g(x) = 2^{-x}$



$y = 2^x$

$\Longrightarrow$



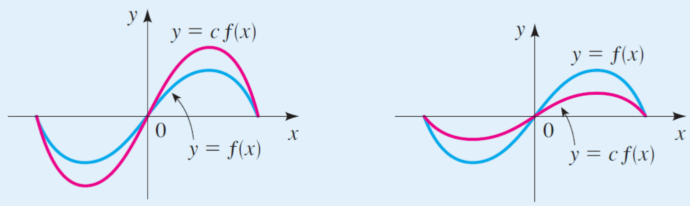
$g(x) = 2^{-x}$

Vertical Stretching and Shrinking of Graphs

To graph $y = cf(x)$:

If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.

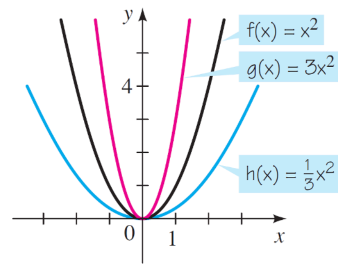


$c > 1$ $\qquad$ $0 < c < 1$

EXAMPLE: Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = 3x^2$

(b) $h(x) = \dfrac{1}{3}x^2$

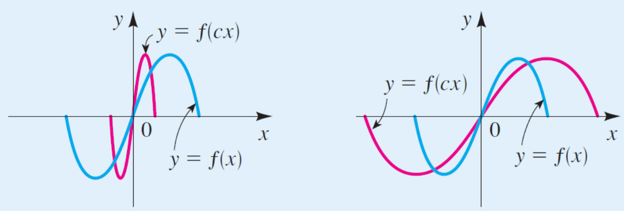


Horizontal Shrinking and Stretching of Graphs

To graph $y = f(cx)$:

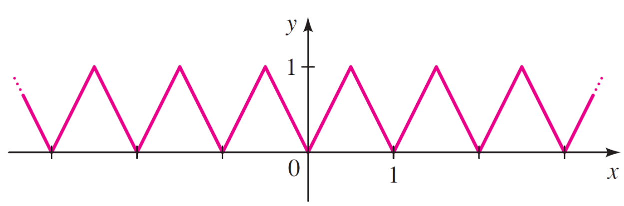
If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.



$c > 1$ $\qquad$ $0 < c < 1$

EXAMPLE: The graph of $y = f(x)$ is shown below.



Sketch the graph of each function.

(a) $y = f(2x)$ $\qquad$ (b) $y = f\left(\frac{1}{2}x\right)$