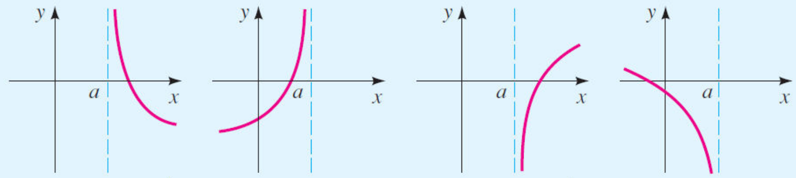
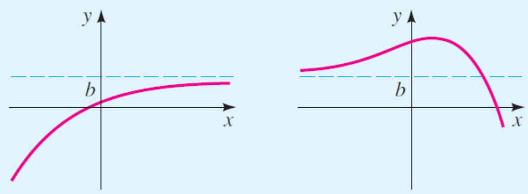
Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or left.



$y \to \infty$ as $x \to a^+$ $\quad$ $y \to \infty$ as $x \to a^-$ $\quad$ $y \to -\infty$ as $x \to a^+$ $\quad$ $y \to -\infty$ as $x \to a^-$

2. The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.



$y \to b$ as $x \to \infty$ $\quad$ $y \to b$ as $x \to -\infty$

EXAMPLE: Sketch a graph of the rational function $f(x) = \dfrac{1}{x}$.

Solution: First note that the function $f(x) = \dfrac{1}{x}$ is not defined for $x = 0$. The tables below show the behavior of f near zero.

$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

x	$f(x)$
-0.1	-10
-0.01	-100
-0.00001	-100,000

x	$f(x)$
0.1	10
0.01	100
0.00001	100,000

The next two tables show how $f(x)$ changes as $|x|$ becomes large.

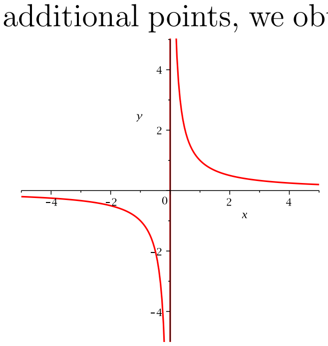
$$\frac{1}{\text{BIG NUMBER}} = \text{small number}$$

x	$f(x)$
-10	-0.1
-100	-0.01
-100,000	-0.00001

x	$f(x)$
10	0.1
100	0.01
100,000	0.00001

Using the information in these tables and plotting a few additional points, we obtain the graph.

x	$f(x) = \frac{1}{x}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$



The line $x = 0$ that is the vertical asymptote. The line $y = 0$ is the horizontal asymptote.

EXAMPLE: Sketch a graph of the rational function $f(x) = \dfrac{1}{x+5}$.

Solution: First note that the function $f(x) = \dfrac{1}{x+5}$ is not defined for $x = -5$. The tables below show the behavior of f near -5.

$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

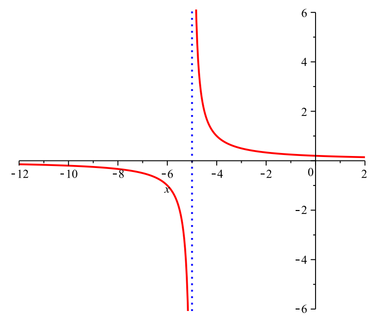
x	$x+5$	$f(x)$
-5.1	-0.1	-10
-5.01	-0.01	-100
-5.00001	-0.00001	-100,000

x	$x+5$	$f(x)$
-4.9	0.1	10
-4.99	0.01	100
-4.99999	0.00001	100,000

As $|x|$ becomes larger and larger, so does the absolute value of the denominator $x+5$. Hence, $f(x) = \dfrac{1}{x+5}$ gets closer and closer to 0.

Using the information in these tables and plotting a few additional points, we obtain the graph.

The line $x = -5$ that is the vertical asymptote. The line $y = 0$ is the horizontal asymptote.



EXAMPLE: Sketch a graph of the rational function $f(x) = \dfrac{x}{1-x}$.

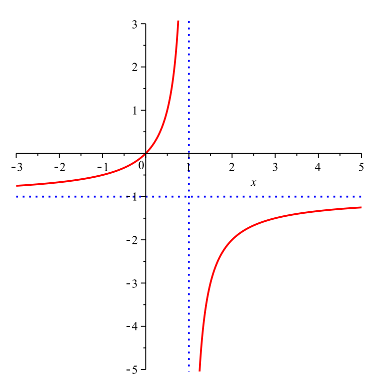
Solution: Find the vertical asymptote by setting the denominator equal to 0 and then solving for x:

$$1 - x = 0 \quad \Longrightarrow \quad x = 1$$

In order to see what the graph looks like when x is very large, we rewrite the rule of the function. When $x \neq 0$, dividing both numerator and denominator by x does not change the value of the function:

$$f(x) = \frac{x}{1-x} = \frac{\frac{x}{x}}{\frac{1-x}{x}} = \frac{\frac{x}{x}}{\frac{1}{x} - \frac{x}{x}} = \frac{1}{\frac{1}{x} - 1}$$

Now, when $|x|$ is very large, the fraction $1/x$ is very close to 0. Therefore, the denominator of $f(x)$ is very close to $0 - 1 = -1$. Hence, $f(x)$ is very close to $1/(-1) = -1$ when $|x|$ is large, so the line $y = -1$ is the horizontal asymptote of the graph, as shown in the Figure below.



EXAMPLE: Sketch a graph of the rational function $f(x) = \dfrac{5x+7}{3x+6}$.

Solution: Find the vertical asymptote by setting the denominator equal to 0 and then solving for x:

$$3x + 6 = 0$$

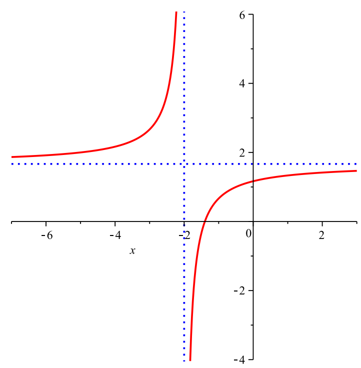
$$3x = -6$$

$$x = \frac{-6}{3} = -2$$

In order to see what the graph looks like when x is very large, we rewrite the rule of the function. When $x \neq 0$, dividing both numerator and denominator by x does not change the value of the function:

$$f(x) = \frac{5x+7}{3x+6} = \frac{\dfrac{5x+7}{x}}{\dfrac{3x+6}{x}} = \frac{\dfrac{5x}{x} + \dfrac{7}{x}}{\dfrac{3x}{x} + \dfrac{6}{x}} = \frac{5 + \dfrac{7}{x}}{3 + \dfrac{6}{x}}$$

Now, when $|x|$ is very large, the fractions $7/x$ and $6/x$ are very close to 0. Therefore, the numerator of $f(x)$ is very close to $5 + 0 = 5$ and the denominator is very c lose to $3 + 0 = 3$. Hence, $f(x)$ is very close to $5/3$ when $|x|$ is large, so the line $y = 5/3$ is the horizontal asymptote of the graph, as shown in the Figure above (right).



Asymptotes of Rational Functions

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

1. The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.

2. (a) If $n < m$, then r has horizontal asymptote $y = 0$.

(b) If $n = m$, then r has horizontal asymptote $y = \dfrac{a_n}{b_m}$.

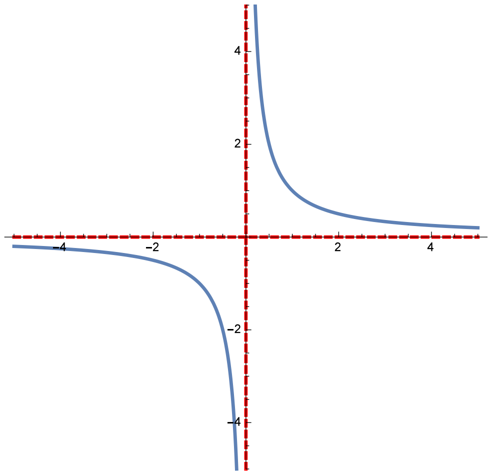
(c) If $n > m$, then r has no horizontal asymptote.

EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{1}{x}$.

Solution:

Vertical Asymptotes: The line $x = 0$ is the vertical asymptote because the denominator of r is zero and the numerator is nonzero when $x = 0$.

Horizontal Asymptote: The degree of the numerator is less than the degree of the denominator, therefore the horizontal asymptote is the line $y = 0$.

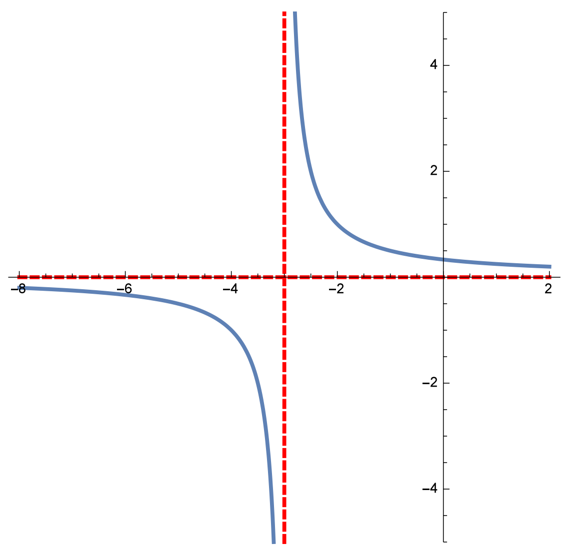


EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{1}{x+3}$.

Solution:

Vertical Asymptotes: The line $x = -3$ is the vertical asymptote because the denominator of r is zero and the numerator is nonzero when $x = -3$.

Horizontal Asymptote: The degree of the numerator is less than the degree of the denominator, therefore the horizontal asymptote is the line $y = 0$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{x}{x-2}$.

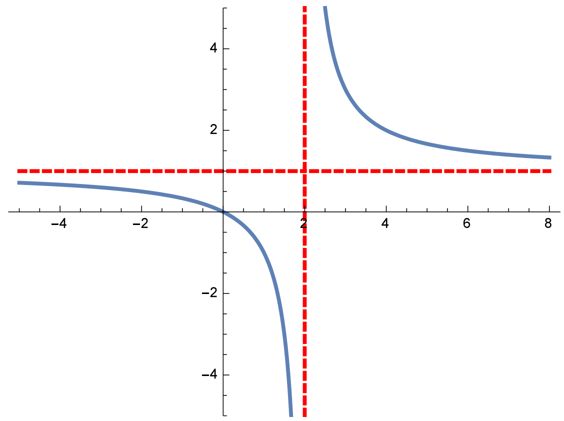
Solution:

Vertical Asymptotes: The line $x = 2$ is the vertical asymptote because the denominator of r is zero and the numerator is nonzero when $x = 2$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{1} = 1$$

Thus, the horizontal asymptote is the line $y = 1$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{5x-1}{2x+3}$.

Solution:

Vertical Asymptotes: The line $x = -3/2$ is the vertical asymptote because the denominator of r is zero and the numerator is nonzero when $x = -3/2$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{5}{2}$$

Thus, the horizontal asymptote is the line $y = \dfrac{5}{2}$.