Section 3.6 Rational Functions

DEFINITION: A **rational function** is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials with $Q(x) \neq 0$.

EXAMPLE: $f(x) = \frac{1}{x}$, $g(x) = -\frac{7}{4x+3}$, $h(x) = \frac{x^2+5x+11}{2x^4+x^3-5x^2+2x-5}$, etc.

Linear Rational Functions

Linear rational functions are rational functions in which both numerator and denominator are first-degree or constant polynomials.

EXAMPLE: Sketch a graph of the rational function $f(x) = \frac{1}{x}$.

Solution: First note that the function $f(x) = \frac{1}{x}$ is not defined for $x = 0$. The tables below show the behavior of f near zero.

$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

x	$f(x)$
-0.1	-10
-0.01	-100
-0.00001	-100,000

x	$f(x)$
0.1	10
0.01	100
0.00001	100,000

The next two tables show how $f(x)$ changes as $|x|$ becomes large.

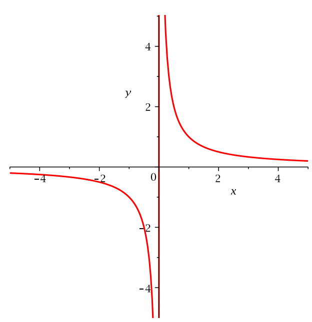
$$\frac{1}{\text{BIG NUMBER}} = \text{small number}$$

x	$f(x)$
-10	-0.1
-100	-0.01
-100,000	-0.00001

x	$f(x)$
10	0.1
100	0.01
100,000	0.00001

Using the information in these tables and plotting a few additional points, we obtain the graph.

x	$f(x) = \frac{1}{x}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$



The vertical line $x = 0$ that is approached by the curve is called a *vertical asymptote*. The horizontal line $y = 0$ is called a *horizontal asymptote*.

EXAMPLE: Sketch a graph of the rational function $f(x)=\dfrac{1}{x+5}$.

Solution: First note that the function $f(x)=\dfrac{1}{x+5}$ is not defined for $x=-5$. The tables below show the behavior of f near -5.

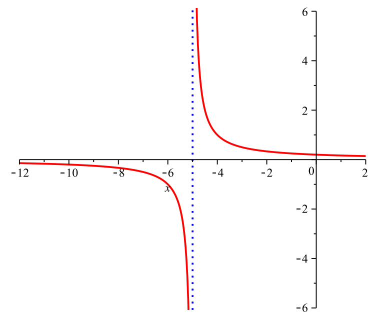
$$\frac{1}{\text{small number}}=\text{BIG NUMBER}$$

x	$x+5$	$f(x)$
-5.1	-0.1	-10
-5.01	-0.01	-100
-5.00001	-0.00001	-100,000

x	$x+5$	$f(x)$
-4.9	0.1	10
-4.99	0.01	100
-4.99999	0.00001	100,000

As $|x|$ becomes larger and larger, so does the absolute value of the denominator $x+5$. Hence, $f(x)=\dfrac{1}{x+5}$ gets closer and closer to 0.

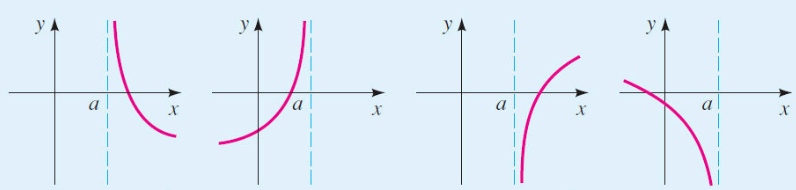
Using the information in these tables and plotting a few additional points, we obtain the graph.



The vertical line $x=-5$ that is approached by the curve is called a *vertical asymptote*. The horizontal line $y=0$ is called a *horizontal asymptote*. Informally speaking, an asymptote of a function is a line that the graph of the function gets closer and closer to as one travels along that line.

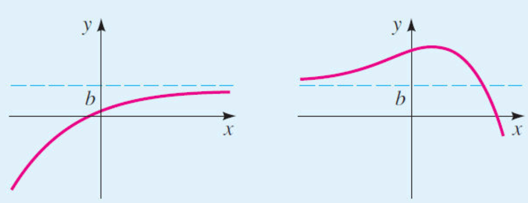
Definition of Vertical and Horizontal Asymptotes

1. The line $x=a$ is a **vertical asymptote** of the function $y=f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or left.



$y\to\infty$ as $x\to a^+$ $\quad$ $y\to\infty$ as $x\to a^-$ $\quad$ $y\to-\infty$ as $x\to a^+$ $\quad$ $y\to-\infty$ as $x\to a^-$

2. The line $y=b$ is a **horizontal asymptote** of the function $y=f(x)$ if y approaches b as x approaches $\pm\infty$.



$y\to b$ as $x\to\infty$ $\quad$ $y\to b$ as $x\to-\infty$

EXAMPLE: Sketch a graph of the rational function $f(x) = \dfrac{x}{1-x}$.

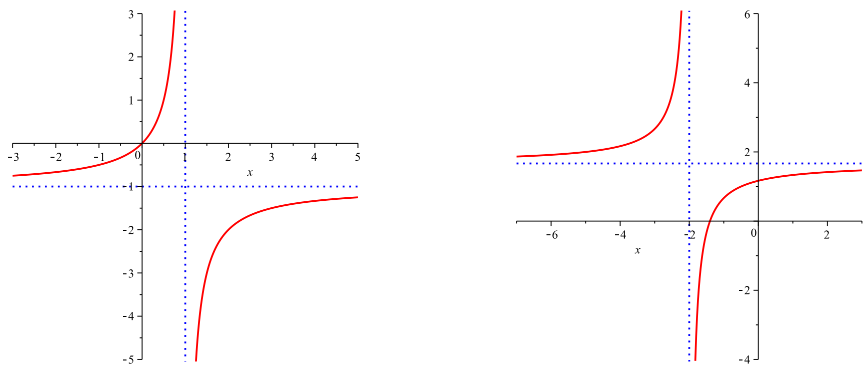
Solution: Find the vertical asymptote by setting the denominator equal to 0 and then solving for x:

$$1 - x = 0 \quad \Longrightarrow \quad x = 1$$

In order to see what the graph looks like when x is very large, we rewrite the rule of the function. When $x \neq 0$, dividing both numerator and denominator by x does not change the value of the function:

$$f(x) = \frac{x}{1-x} = \frac{\dfrac{x}{x}}{\dfrac{1-x}{x}} = \frac{\dfrac{x}{x}}{\dfrac{1}{x} - \dfrac{x}{x}} = \frac{1}{\dfrac{1}{x} - 1}$$

Now, when $|x|$ is very large, the fraction $1/x$ is very close to 0. Therefore, the denominator of $f(x)$ is very close to $0 - 1 = -1$. Hence, $f(x)$ is very close to $1/(-1) = -1$ when $|x|$ is large, so the line $y = -1$ is the horizontal asymptote of the graph, as shown in the Figure below (left).



EXAMPLE: Sketch a graph of the rational function $f(x) = \dfrac{5x+7}{3x+6}$.

Solution: Find the vertical asymptote by setting the denominator equal to 0 and then solving for x:

$$3x + 6 = 0$$

$$3x = -6$$

$$x = \frac{-6}{3} = -2$$

In order to see what the graph looks like when x is very large, we rewrite the rule of the function. When $x \neq 0$, dividing both numerator and denominator by x does not change the value of the function:

$$f(x) = \frac{5x+7}{3x+6} = \frac{\dfrac{5x+7}{x}}{\dfrac{3x+6}{x}} = \frac{\dfrac{5x}{x} + \dfrac{7}{x}}{\dfrac{3x}{x} + \dfrac{6}{x}} = \frac{5 + \dfrac{7}{x}}{3 + \dfrac{6}{x}}$$

Now, when $|x|$ is very large, the fractions $7/x$ and $6/x$ are very close to 0. Therefore, the numerator of $f(x)$ is very close to $5 + 0 = 5$ and the denominator is very c lose to $3 + 0 = 3$. Hence, $f(x)$ is very close to $5/3$ when $|x|$ is large, so the line $y = 5/3$ is the horizontal asymptote of the graph, as shown in the Figure above (right).

Other Rational Functions

EXAMPLE: Sketch a graph of the rational function $f(x) = \dfrac{3x^2}{x^2-5}$.

Solution: Find the vertical asymptote by setting the denominator equal to 0 and then solving for x:

$$\begin{aligned} x^2 &= 5 \\ x^2 - 5 &= 0 \\ x^2 - (\sqrt{5})^2 &= 0 \\ (x+\sqrt{5})(x-\sqrt{5}) &= 0 \\ x+\sqrt{5} = 0 \quad \text{or} \quad x-\sqrt{5} &= 0 \\ x = -\sqrt{5} \quad \text{or} \quad x &= \sqrt{5} \end{aligned}$$

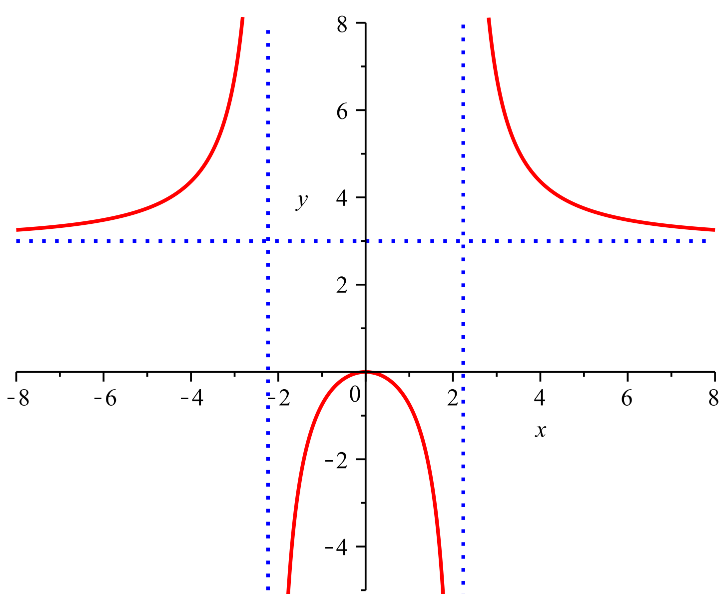
or

$$\begin{aligned} x^2 &= 5 \\ \sqrt{x^2} &= \sqrt{5} \\ |x| &= \sqrt{5} \\ x &= \pm\sqrt{5} \end{aligned}$$

Since neither of these numbers makes the numerator 0 , the lines $x = -\sqrt{5}$ and $x = \sqrt{5}$ are vertical asymptotes of the graph. The horizontal asymptote can be determined by dividing both the numerator and denominator of $f(x)$ by x^2 (the highest power of x that appears in either one):

$$f(x) = \frac{3x^2}{x^2-5} = \frac{\dfrac{3x^2}{x^2}}{\dfrac{x^2-5}{x^2}} = \frac{\dfrac{3x^2}{x^2}}{\dfrac{x^2}{x^2} - \dfrac{5}{x^2}} = \frac{3}{1 - \dfrac{5}{x^2}}$$

When $|x|$ is very large, the fraction $5/x^2$ is very close to 0, so the denominator is very close to 1 and $f(x)$ is very close to 3. Hence, the line $y = 3$ is the horizontal asymptote of the graph. Using this information and plotting several points in each of the three regions determined by the vertical asymptotes, we obtain the Figure below.



Asymptotes of Rational Functions

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

1. The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.

2. (a) If $n < m$, then r has horizontal asymptote $y = 0$.

(b) If $n = m$, then r has horizontal asymptote $y = \frac{a_n}{b_m}$.

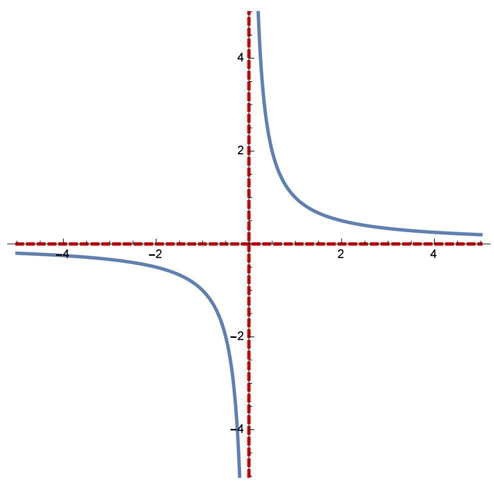
(c) If $n > m$, then r has no horizontal asymptote.

EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \frac{1}{x}$.

Solution:

Vertical Asymptotes: The line $x = 0$ is the vertical asymptote because the denominator of r is zero and the numerator is nonzero when $x = 0$.

Horizontal Asymptote: The degree of the numerator is less than the degree of the denominator, therefore the horizontal asymptote is the line $y = 0$.



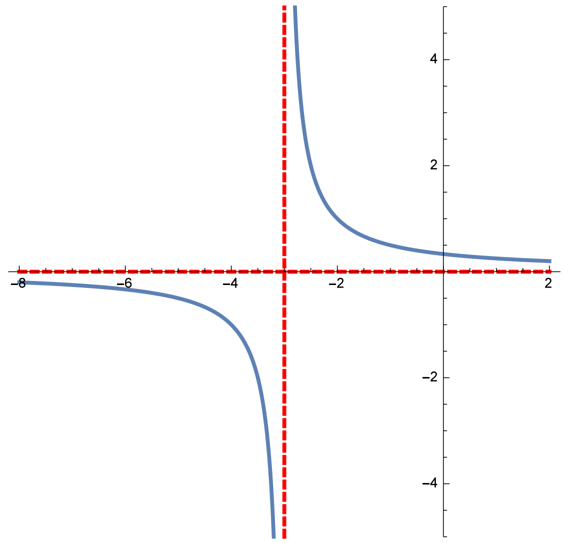
EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \frac{1}{x+3}$.

EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{1}{x+3}$.

Solution:

Vertical Asymptotes: The line $x = -3$ is the vertical asymptote because the denominator of r is zero and the numerator is nonzero when $x = -3$.

Horizontal Asymptote: The degree of the numerator is less than the degree of the denominator, therefore the horizontal asymptote is the line $y = 0$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{x}{x-2}$.

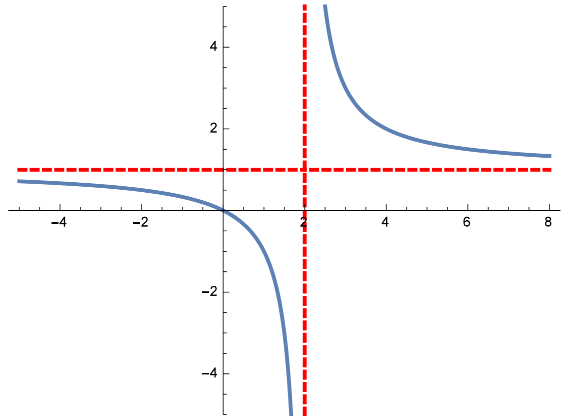
Solution:

Vertical Asymptotes: The line $x = 2$ is the vertical asymptote because the denominator of r is zero and the numerator is nonzero when $x = 2$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{1} = 1$$

Thus, the horizontal asymptote is the line $y = 1$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{5x-1}{2x+3}$.

EXAMPLE: Find the vertical and horizontal asymptotes of $r(x)=\dfrac{5x-1}{2x+3}$.

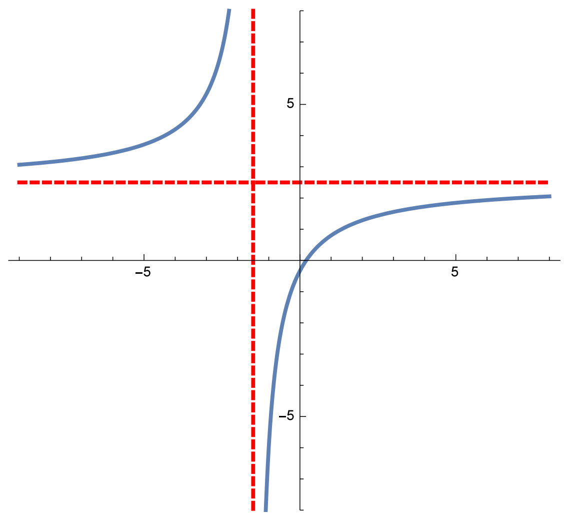
Solution:

Vertical Asymptotes: The line $x=-3/2$ is the vertical asymptote because the denominator of r is zero and the numerator is nonzero when $x=-3/2$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}=\frac{5}{2}$$

Thus, the horizontal asymptote is the line $y=\dfrac{5}{2}$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x)=\dfrac{3x^2-2x-1}{2x^2+3x-2}$.

Solution:

Vertical Asymptotes: We first factor

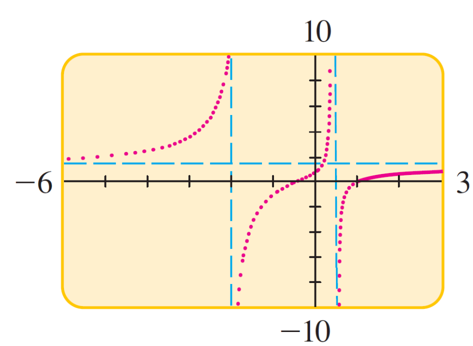
$$r(x)=\frac{3x^2-2x-1}{2x^2+3x-2}=\frac{(3x+1)(x-1)}{(2x-1)(x+2)}$$

The lines $x=\dfrac{1}{2}$ and $x=-2$ are vertical asymptotes because the denominator of r is zero and the numerator is nonzero when $x=\dfrac{1}{2}$ or $x=-2$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}=\frac{3}{2}$$

Thus, the horizontal asymptote is the line $y=\dfrac{3}{2}$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x)=\dfrac{x^2-4x+4}{9x^2-9x+2}$.

Solution:

Vertical Asymptotes: We first factor

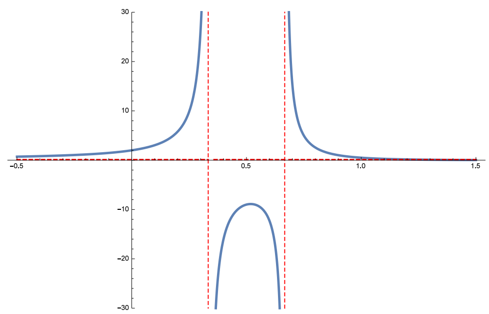
$$r(x)=\frac{x^2-4x+4}{9x^2-9x+2}=\frac{(x-2)^2}{(3x-1)(3x-2)}$$

The lines $x=\dfrac{1}{3}$ and $x=\dfrac{2}{3}$ are vertical asymptotes because the denominator of r is zero and the numerator is nonzero when $x=\dfrac{1}{3}$ or $x=\dfrac{2}{3}$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}=\frac{1}{9}$$

Thus, the horizontal asymptote is the line $y=\dfrac{1}{9}$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x)=\dfrac{x^2-x-2}{x^2+2x-8}$.

Solution:

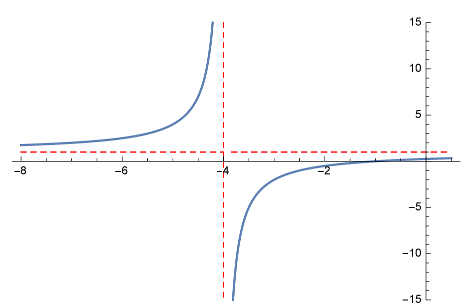
Vertical Asymptote: We first factor

$$r(x)=\frac{x^2-x-2}{x^2+2x-8}=\frac{(x-2)(x+1)}{(x-2)(x+4)}$$

It follows that

$$r(x)=\frac{x+1}{x+4}$$

if $x\neq 2$. The line $x=-4$ is the vertical asymptote because the denominator $x+4$ is zero and the numerator is nonzero when $x=-4$. The line $x=2$ is *not* the vertical asymptote because the denominator $x+4$ is nonzero when $x=2$.



Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}=\frac{1}{1}=1$$

Thus, the horizontal asymptote is the line $y=1$.

Applications

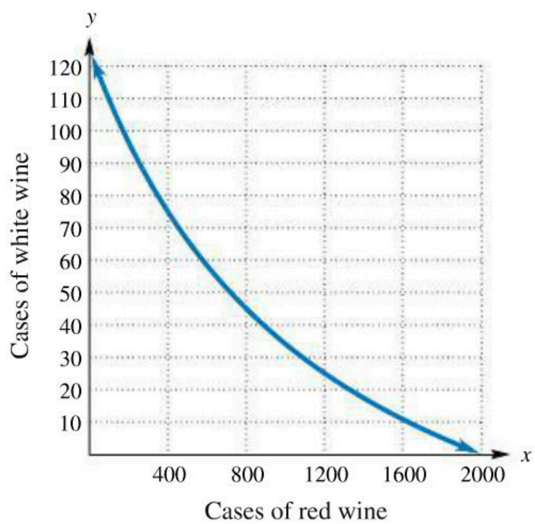
In management, **product-exchange functions** give the relationship between quantities of two items that can be produced by the same machine or factory. For example, an oil refinery can produce gasoline, heating oil, or a combination of the two; a winery can produce red wine, white wine, or a combination of the two. The next example discusses a product-exchange function.

EXAMPLE: The product-exchange function for the Fruits of the Earth Winery for red wine x and white wine y, in number of cases, is

$$y = \frac{150,000 - 75x}{1200 + x}$$

Graph the function and find the maximum quantity of each kind of wine that can be produced.

Solution: Only nonnegative values of x and y make sense in this situation, so we graph the function in the first quadrant (see the Figure below). Note that the y-intercept of the graph (found by setting $x = 0$) is 125 and the x-intercept (found by setting $y = 0$ and solving for x) is 2000. Since we are interested only in the portion of the graph in Quadrant I, we can find a few more points in that quadrant and complete the graph as shown in the Figure below.



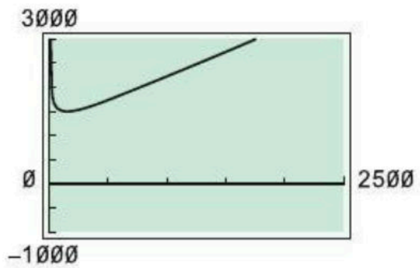
The maximum value of y occurs when $x = 0$, so the maximum amount of white wine that can be produced is 125 cases, as given by the y-intercept. The x-intercept gives the maximum amount of red wine that can be produced: 2000 cases.

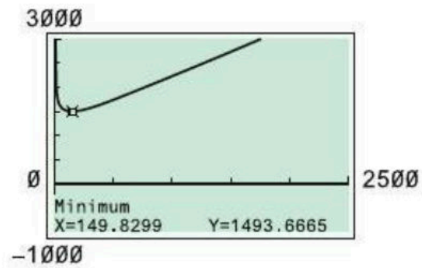
EXAMPLE: A retailer buys 2500 specialty lightbulbs from a distributor each year. In addition to the cost of each bulb, there is a fee for each order, so she wants to order as few times as possible. However, storage costs are higher when there are fewer orders (and hence more bulbs per order to store). Past experience shows that the total annual cost (for the bulbs, ordering fees, and storage costs) is given by the rational function.

$$C(x) = \frac{.98x^2 + 1200x + 22,000}{x}$$

where x is the number of bulbs ordered each time. How many bulbs should be ordered each time in order to have the smallest possible cost?

Solution: Graph the cost function $C(x)$ in a window with $0 \leq x \leq 2500$ (because the retailer cannot order a negative number of bulbs and needs only 2500 for the year).





For each point on the graph in the Figure above (left)

> the x-coordinate is the number of bulbs ordered each time
>
> the y-coordinate is the annual cost when x bulbs are ordered each time.

Use the minimum finder on a graphing calculator to find the point with the smallest y-coordinate, which is approximately $(149.83, 1493.67)$, as shown in the Figure above (right). Since the retailer cannot order part of a lightbulb, she should order 150 bulbs each time, for an approximate annual cost of \$1494.