Other Rational Functions

EXAMPLE: Sketch a graph of the rational function $f(x) = \dfrac{3x^2}{x^2 - 5}$.

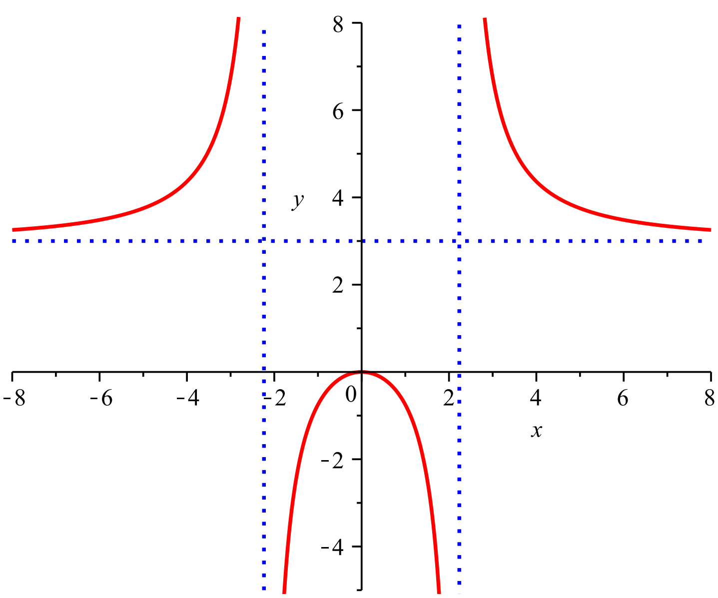
Solution: Find the vertical asymptote by setting the denominator equal to 0 and then solving for x:

$$\begin{aligned} x^2 &= 5 \\ x^2 - 5 &= 0 \\ x^2 - (\sqrt{5})^2 &= 0 \\ (x+\sqrt{5})(x-\sqrt{5}) &= 0 \\ x+\sqrt{5} = 0 \quad \text{or} \quad x - \sqrt{5} &= 0 \\ x = -\sqrt{5} \quad \text{or} \quad x &= \sqrt{5} \end{aligned} \qquad \text{or} \qquad \begin{aligned} x^2 &= 5 \\ \sqrt{x^2} &= \sqrt{5} \\ |x| &= \sqrt{5} \\ x &= \pm\sqrt{5} \end{aligned}$$

Since neither of these numbers makes the numerator 0 , the lines $x = -\sqrt{5}$ and $x = \sqrt{5}$ are vertical asymptotes of the graph. The horizontal asymptote can be determined by dividing both the numerator and denominator of $f(x)$ by x^2 (the highest power of x that appears in either one):

$$f(x) = \frac{3x^2}{x^2 - 5} = \frac{\dfrac{3x^2}{x^2}}{\dfrac{x^2 - 5}{x^2}} = \frac{\dfrac{3x^2}{x^2}}{\dfrac{x^2}{x^2} - \dfrac{5}{x^2}} = \frac{3}{1 - \dfrac{5}{x^2}}$$

When $|x|$ is very large, the fraction $5/x^2$ is very close to 0, so the denominator is very close to 1 and $f(x)$ is very close to 3. Hence, the line $y = 3$ is the horizontal asymptote of the graph. Using this information and plotting several points in each of the three regions determined by the vertical asymptotes, we obtain the Figure below.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$.

Solution:

Vertical Asymptotes: We first factor

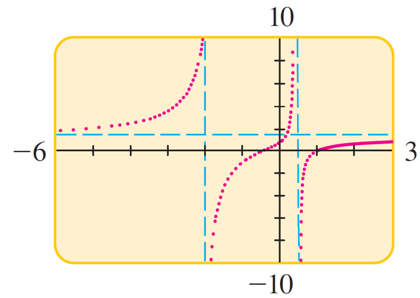
$$r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2} = \frac{(3x+1)(x-1)}{(2x-1)(x+2)}$$

The lines $x = \dfrac{1}{2}$ and $x = -2$ are vertical asymptotes because the denominator of r is zero and the numerator is nonzero when $x = \dfrac{1}{2}$ or $x = -2$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{3}{2}$$

Thus, the horizontal asymptote is the line $y = \dfrac{3}{2}$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{x^2 - 4x + 4}{9x^2 - 9x + 2}$.

Solution:

Vertical Asymptotes: We first factor

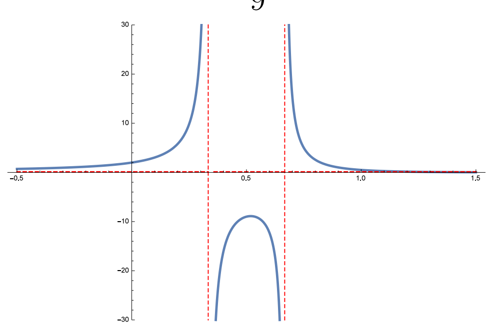
$$r(x) = \frac{x^2 - 4x + 4}{9x^2 - 9x + 2} = \frac{(x-2)^2}{(3x-1)(3x-2)}$$

The lines $x = \dfrac{1}{3}$ and $x = \dfrac{2}{3}$ are vertical asymptotes because the denominator of r is zero and the numerator is nonzero when $x = \dfrac{1}{3}$ or $x = \dfrac{2}{3}$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{9}$$

Thus, the horizontal asymptote is the line $y = \dfrac{1}{9}$.



EXAMPLE: Find the vertical and horizontal asymptotes of $r(x) = \dfrac{x^2 - x - 2}{x^2 + 2x - 8}$.

Solution:

Vertical Asymptote: We first factor

$$r(x) = \frac{x^2 - x - 2}{x^2 + 2x - 8} = \frac{(x-2)(x+1)}{(x-2)(x+4)}$$

It follows that

$$r(x) = \frac{x+1}{x+4}$$

if $x \neq 2$. The line $x = -4$ is the vertical asymptote because the denominator $x + 4$ is zero and the numerator is nonzero when $x = -4$. The line $x = 2$ is *not* the vertical asymptote because the denominator $x + 4$ is nonzero when $x = 2$.

Horizontal Asymptote: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{1} = 1$$

Thus, the horizontal asymptote is the line $y = 1$.