Graphing Techniques

If P is a polynomial function, then c is called a **zero** of P if $P(c) = 0$. In other words, the zeros of P are the solutions of the polynomial equation $P(x) = 0$. Note that if $P(c) = 0$, then the graph of P has an x-intercept at $x = c$, so the x-intercepts of the graph are the zeros of the function.

Real Zeros of Polynomials

If P is a polynomial and c is a real number, then the following are equivalent.

1. c is a zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. $x = c$ is an x-intercept of the graph of P.

Guidelines for Graphing Polynomial Functions

1. **Zeros.** Factor the polynomial to find all its real zeros; these are the x-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

EXAMPLE: Sketch the graph of the polynomial function $P(x) = (x + 2)(x - 1)(x - 3)$.

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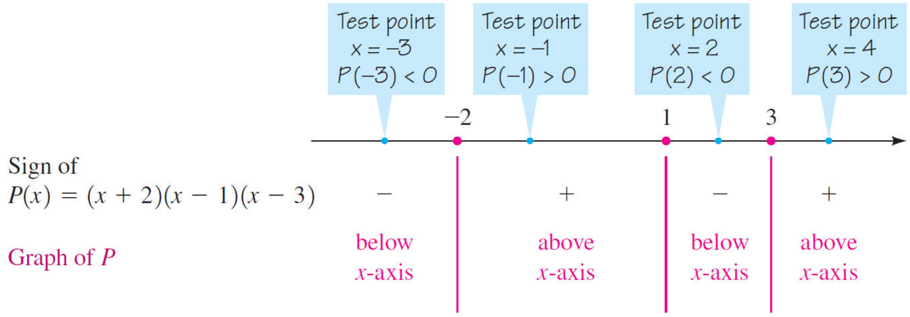
Solution: The zeros are $x = -2, 1$, and 3. These determine the intervals $(-\infty, -2), (-2, 1), (1, 3)$, and $(3, \infty)$. Using test points in these intervals, we get the information in the following sign diagram.



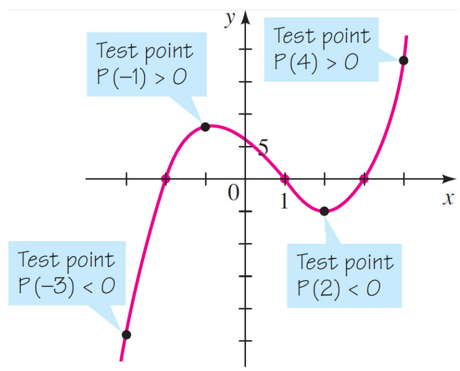
The polynomial P has degree 3 and leading coefficient 1. Thus, P has odd degree and positive leading coefficient, so the end behavior of P is similar to x^3:

$$y \to -\infty \text{ as } x \to -\infty \quad \text{and} \quad y \to \infty \text{ as } x \to \infty$$

DOWN (left) and UP (right)

Plotting a few additional points and connecting them with a smooth curve helps us complete the graph.

	x	$P(x)$
Test point →	−3	−24
	−2	0
Test point →	−1	8
	0	6
	1	0
Test point →	2	−4
	3	0
Test point →	4	18



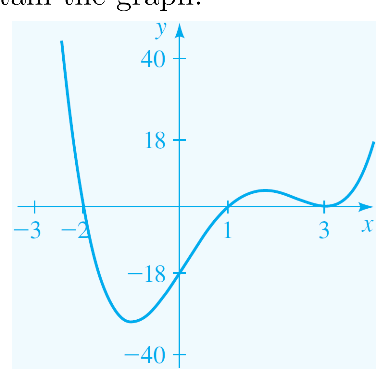
EXAMPLE: Sketch the graph of the polynomial function $P(x) = (x+2)(x-1)(x-3)^2$.

Solution: The zeros are $-2, 1$, and 3. The polynomial P has degree 4 and leading coefficient 1. Thus, P has even degree and positive leading coefficient, so the end behavior of P is similar to x^2:

$$y \to \infty \text{ as } x \to -\infty \quad \text{and} \quad y \to \infty \text{ as } x \to \infty$$

UP (left) and UP (right)

We use test points 0 and 2 to obtain the graph:



EXAMPLE: Let $P(x) = x^3 - 2x^2 - 3x$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution:

(a) To find the zeros, we factor completely:

$$P(x) = x^3 - 2x^2 - 3x = x(x^2 - 2x - 3) = x(x-3)(x+1)$$

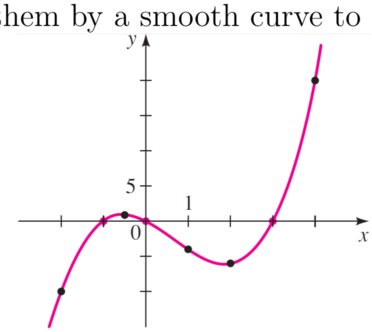
Thus, the zeros are $x = 0$, $x = 3$, and $x = -1$.

(b) The x-intercepts are $x = 0$, $x = 3$, and $x = -1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure we choose test points between (and to the right and left of) successive zeros. The polynomial P has odd degree and positive leading coefficient. Thus, the end behavior of P is similar to x^3:

$$y \to -\infty \text{ as } x \to -\infty \quad \text{and} \quad y \to \infty \text{ as } x \to \infty$$
$$\text{DOWN (left)} \quad \text{and} \quad \text{UP (right)}$$

We plot the points in the table and connect them by a smooth curve to complete the graph.

	x	$P(x)$
	-2	-10
	-1	0
Test point →	$-\frac{1}{2}$	$\frac{7}{8}$
	0	0
Test point →	1	-4
	2	-6
	3	0
	4	20



EXAMPLE: Let $P(x) = x^3 - 9x^2 + 20x$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution:

(a) To find the zeros, we factor completely:

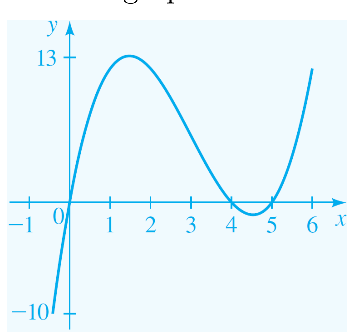
$$P(x) = x^3 - 9x^2 + 20x = x(x^2 - 9x + 20) = x(x-4)(x-5)$$

Thus, the zeros are $x = 0$, $x = 4$, and $x = 5$.

(b) The polynomial P has odd degree and positive leading coefficient. Thus, the end behavior of P is similar to x^3:

$$y \to -\infty \text{ as } x \to -\infty \quad \text{and} \quad y \to \infty \text{ as } x \to \infty$$
$$\text{DOWN (left)} \quad \text{and} \quad \text{UP (right)}$$

We use test points 3 and 4.5 to obtain the graph:



EXAMPLE: Let $P(x) = -2x^4 - x^3 + 3x^2$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution:

(a) To find the zeros, we factor completely:

$$P(x) = -2x^4 - x^3 + 3x^2 = -x^2(2x^2 + x - 3) = -x^2(2x + 3)(x - 1)$$

Thus, the zeros are $x = 0,\ x = -\frac{3}{2}$, and $x = 1$.

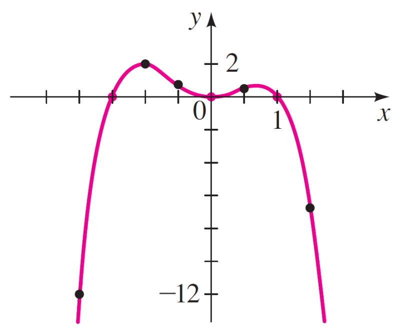
(b) The x-intercepts are $x = 0,\ x = -\frac{3}{2}$, and $x = 1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure we choose test points between (and to the right and left of) successive zeros. The polynomial P has even degree and negative leading coefficient. Thus, the end behavior of P is similar to $-x^2$:

$$y \to -\infty \text{ as } x \to -\infty \quad \text{and} \quad y \to -\infty \text{ as } x \to \infty$$

$$\text{DOWN (left)} \quad \text{and} \quad \text{DOWN (right)}$$

We plot the points in the table and connect them by a smooth curve to complete the graph.

x	$P(x)$
-2	-12
-1.5	0
-1	2
-0.5	0.75
0	0
0.5	0.5
1	0
1.5	-6.75



EXAMPLE: Let $P(x) = 3x^4 - 5x^3 - 12x^2$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution:

(a) To find the zeros, we factor completely:

$$P(x) = 3x^4 - 5x^3 - 12x^2 = x^2(3x^2 - 5x - 12) = x^2(x - 3)(3x + 4)$$

Thus, the zeros are $x = 0,\ x = 3$, and $x = -3/4$.

(b) The polynomial P has even degree and positive leading coefficient. Thus, the end behavior of P is similar to x^2:

$$y \to \infty \text{ as } x \to -\infty \quad \text{and} \quad y \to \infty \text{ as } x \to \infty$$

$$\text{UP (left)} \quad \text{and} \quad \text{UP (right)}$$

We use test points -1 and 1 to obtain the graph.