Section 3.4 Quadratic Functions and Applications

DEFINITION: A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$.

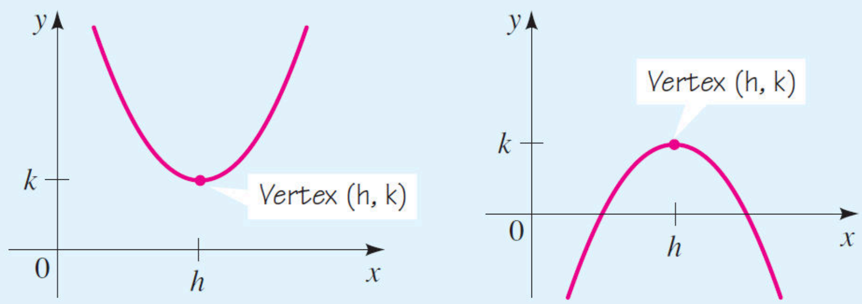
Graphing Quadratic Functions Using the Standard Form

Standard Form of a Quadratic Function

A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of f is a parabola with **vertex** (h, k); the parabola opens upward if $a > 0$ or downward if $a < 0$.



$f(x) = a(x - h)^2 + k,\ a > 0$ $\qquad$ $f(x) = a(x - h)^2 + k,\ a < 0$

EXAMPLE: Let $f(x) = x^2 + 10x - 1$. Express f in standard form. Identify the vertex.

Solution: We have

$$\begin{aligned} f(x) &= x^2 + 10x - 1 \\ &= x^2 + 2x \cdot 5 - 1 \\ &= x^2 + 2x \cdot 5 + 5^2 - 5^2 - 1 \\ &= (x+5)^2 - 26 \\ &= \Big(x - (-5)\Big)^2 + (-26) \end{aligned}$$

The vertex is $(-5, -26)$.

EXAMPLE: Let $f(x) = 2(x - 3)^2 + 5$.

(a) Determine whether the given parabola opens upward or downward, and find its vertex.

(b) Sketch the graph of f.

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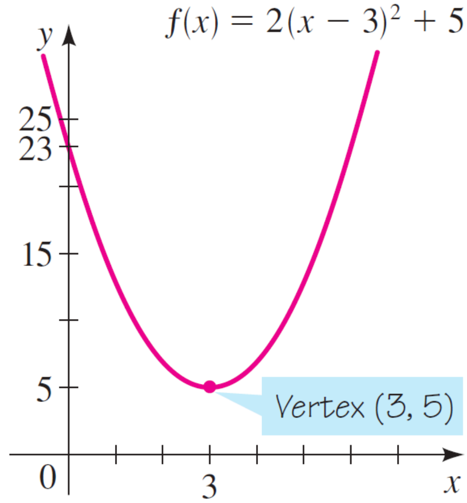
(a) Determine whether the given parabola opens upward or downward, and find its vertex.

(b) Sketch the graph of f.

Solution:

(a) Since $a = 2 > 0$, the parabola opens upward. The vertex of the parabola is at $(3, 5)$.

(b) The graph is shown below.

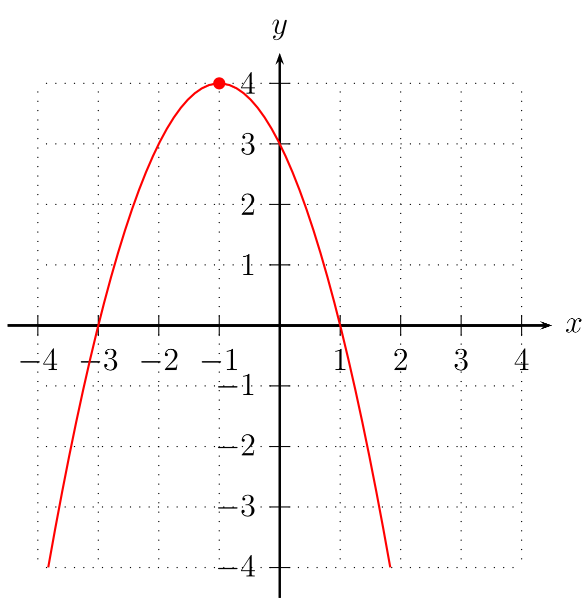


EXAMPLE: Let $f(x) = -(x+1)^2 + 4$. Determine whether the given parabola opens upward or downward, and find its vertex. Then sketch the graph of f.

Solution: Since $a = -1 < 0$, the parabola opens downward. The vertex of the parabola is at $(-1, 4)$, because

$$f(x) = -(x+1)^2 + 4 = -\Big(x - (-1)\Big)^2 + 4 \quad \Longrightarrow \quad h = -1, \ k = 4$$

The graph is shown below.



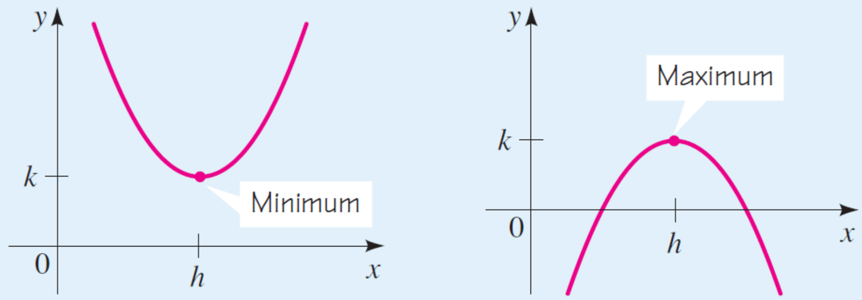
Maximum and Minimum Values of Quadratic Functions

Maximum or Minimum Value of a Quadratic Function

Let f be a quadratic function with standard form $f(x) = a(x - h)^2 + k$. The maximum or minimum value of f occurs at $x = h$.

If $a > 0$, then the **minimum value** of f is $f(h) = k$.

If $a < 0$, then the **maximum value** of f is $f(h) = k$.



$f(x) = a(x - h)^2 + k, a > 0$ $\qquad$ $f(x) = a(x - h)^2 + k, a < 0$

EXAMPLE: Consider the quadratic function $f(x) = 5(x - 3)^2 + 4$.

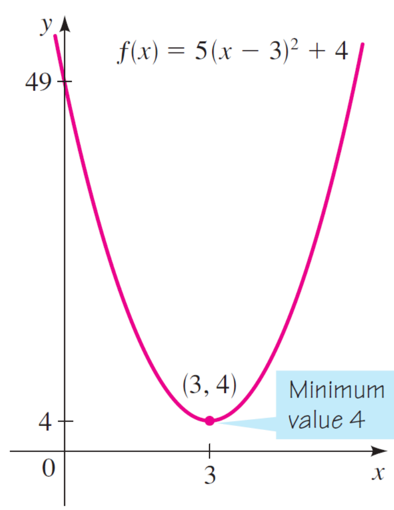
(a) Find the maximum or minimum value of f.

(b) Sketch the graph of f.

Solution:

(a) Since the coefficient of x^2 is positive, f has a *minimum* value. The minimum value is $f(3) = 4$.

(b) The graph is a parabola that has its vertex at $(3, 4)$ and opens upward, as sketched in the Figure below.



EXAMPLE: Consider the quadratic function $f(x) = 2(x - 2)^2 + 5$. Find the maximum or minimum value of f. Then sketch the graph of f.

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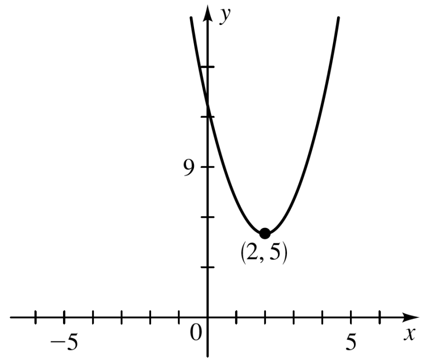
Solution: Since the coefficient of x^2 is positive, f has a *minimum* value. The minimum value is $f(2) = 5$.

The graph is a parabola that has its vertex at $(2, 5)$ and opens upward, as sketched in the Figure on the right.



EXAMPLE: Consider the quadratic function $f(x) = -\left(x - \frac{1}{2}\right)^2 + \frac{9}{4}$.

(a) Find the maximum or minimum value of f.

(b) Find the x- and y-intercepts of f.

(c) Sketch the graph of f.

Solution:

(a) Since the coefficient of x^2 is negative, f has a *maximum* value, which is $f\left(\frac{1}{2}\right) = \frac{9}{4}$.

(b) The y-intercept is $(0, 2)$, since

$$f(0) = -\left(0 - \frac{1}{2}\right)^2 + \frac{9}{4} = -\left(-\frac{1}{2}\right)^2 + \frac{9}{4} = -\frac{1}{4} + \frac{9}{4} = \frac{-1+9}{4} = \frac{8}{4} = 2$$

To find the x-intercepts, we set $f(x) = 0$:

$$-\left(x - \frac{1}{2}\right)^2 + \frac{9}{4} = 0$$

$$\frac{9}{4} = \left(x - \frac{1}{2}\right)^2$$

$$\sqrt{\frac{9}{4}} = \sqrt{\left(x - \frac{1}{2}\right)^2}$$

$$\frac{3}{2} = \left|x - \frac{1}{2}\right|$$

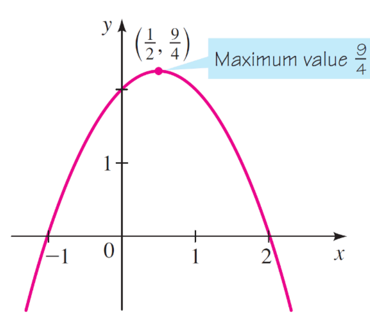
$$\pm\frac{3}{2} = x - \frac{1}{2}$$

therefore

$$x = \frac{1}{2} \pm \frac{3}{2} = \frac{1 \pm 3}{2} \quad \Longrightarrow \quad x = -1,\ 2$$

Thus, the x-intercepts are $(-1, 0)$ and $(2, 0)$.

(c) The graph of f is sketched in the Figure on the right.



Maximum or Minimum Value of a Quadratic Function

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\frac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\frac{b}{2a}\right)$.

EXAMPLE: Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$ (b) $g(x) = -2x^2 + 4x - 5$

Solution:

(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus, the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value

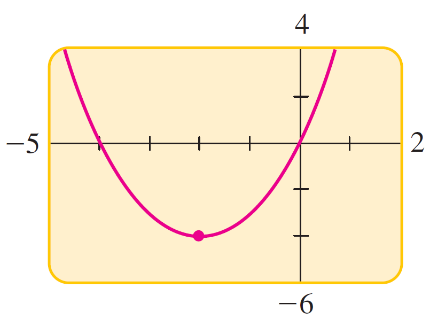
$$f(-2) = (-2)^2 + 4(-2) = -4$$

(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus, the maximum or minimum value occurs at

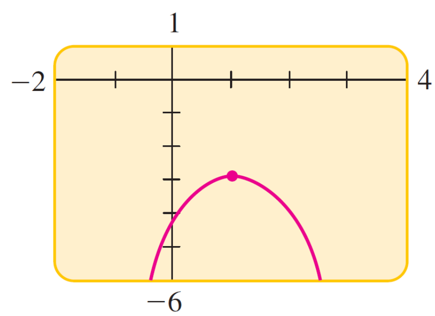
$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot (-2)} = 1$$

Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$



The minimum value occurs at $x = -2$.



The maximum value occurs at $x = 1$.

Applications

EXAMPLE: Most cars get their best gas mileage when traveling at a relatively modest speed. The gas mileage M for a certain new car is modeled by the function

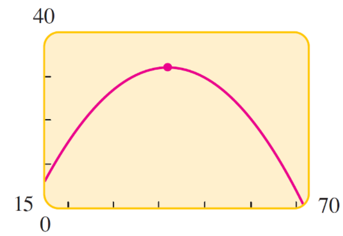
$$M(s) = -\frac{1}{28}s^2 + 3s - 31, \quad 15 \le s \le 70$$

where s is the speed in mi/h and M is measured in mi/gal. What is the car's best gas mileage, and at what speed is it attained?

Solution: The function M is a quadratic function with $a = -\frac{1}{28}$ and $b = 3$. Thus, its maximum value occurs when

$$s = -\frac{b}{2a} = -\frac{3}{2\left(-\frac{1}{28}\right)} = -\frac{3 \cdot 28}{2\left(-\frac{1}{28}\right) \cdot 28} = -\frac{3 \cdot 28}{2(-1)} = -\frac{3 \cdot 14}{(-1)} = 42$$

The maximum is $M(42) = -\frac{1}{28}(42)^2 + 3(42) - 31 = 32$. So the car's best gas mileage is 32 mi/gal, when it is traveling at 42 mi/h.



The maximum gas mileage occurs at 42 mi/h.

EXAMPLE: Lynn Wolf owns and operates Wolf's microbrewery. She has hired a consultant to analyze her business operations. The consultant tells her that her daily profits from the sale of x cases of beer are given by

$$P(x) = -x^2 + 120x$$

Find the vertex, determine if it is a maximum or minimum, write the equation of the axis of the parabola, and compute the x- and y-intercepts of the profit function $P(x)$.

Solution: Since $a = -1$ and $b = 120$, the x-value of the vertex is

$$x = -\frac{b}{2a} = -\frac{120}{2(-1)} = 60$$

The y-value of the vertex is

$$P(60) = -(60)^2 + 120(60) = 3600$$

The vertex is $(60, 3600)$ and since a is negative, it is a maximum because the parabola opens downward. The axis of the parabola is $x = 60$. The intercepts are found by setting x and y equal to 0.

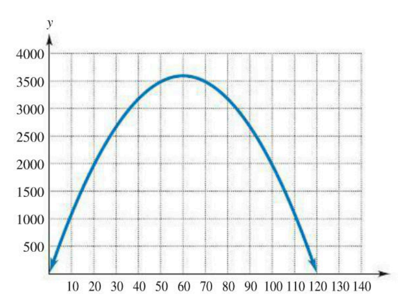
To find the x-intercepts, we set $f(x) = 0$ and factor the resulting equation.

$$-x^2 + 120x = 0$$

$$-x(x - 120) = 0$$

$$-x = 0 \quad \text{or} \quad x - 120 = 0$$

$$x = 0 \quad \text{or} \quad x = 120$$



The x- intercepts are $(0, 0)$ and $(120, 0)$.

The y-intercept is $(0, 0)$, since $f(0) = -0^2 + 120(0) = 0$.

EXAMPLE: Suppose that the price of and demand for an item are related by

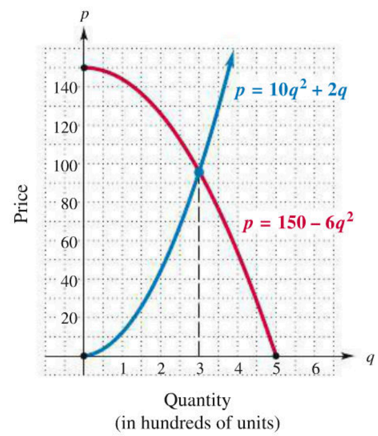
$$p = 150 - 6q^2 \qquad \text{(Demand function)}$$

where p is the price (in dollars) and q is the number of items demanded (in hundreds). Suppose also that the price and supply are related by

$$p = 10q^2 + 2q \qquad \text{(Supply function)}$$

where q is the number of items supplied (in hundreds). Find the equilibrium quantity and the equilibrium price.

Solution: The graphs of both of these equations are parabolas (see the Figure on the right). Only those portions of the graphs which lie in the first quadrant are included, because none of supply, demand, or price can be negative. The point where the demand and supply curves intersect is the equilibrium point. Its first coordinate is the equilibrium quantity, and its second coordinate is the equilibrium price. These coordinates may be found in two ways.



Algebraic Method: At the equilibrium point, the second coordinate of the demand curve must be the same as the second coordinate of the supply curve, so that

$$\begin{aligned} 150 - 6q^2 &= 10q^2 + 2q \\ 0 &= 10q^2 + 2q - 150 + 6q^2 \\ 0 &= 16q^2 + 2q - 150 \\ 0 &= 8q^2 + q - 75 \end{aligned}$$

This equation can be solved by the Quadratic Formula. Here, $a = 8$, $b = 1$, and $c = -75$:

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(8)(-75)}}{2(8)} = \frac{-1 \pm \sqrt{1 + 2400}}{16} = \frac{-1 \pm \sqrt{2401}}{16} = \frac{-1 \pm 49}{16}$$

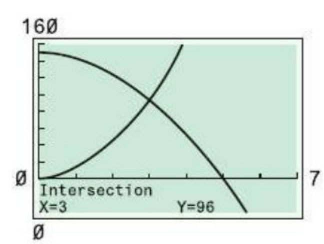
so

$$q = \frac{-1 + 49}{16} = \frac{48}{16} = 3 \qquad \text{or} \qquad q = \frac{-1 - 49}{16} = -\frac{50}{16} = -\frac{25}{8}$$

It is not possible to make $-25/8$ units, so discard that answer and use only $q = 3$. Hence, the equilibrium quantity is 300. Find the equilibrium price by substituting 3 for q in either the supply or the demand function (and check your answer by using the other one). Using the supply function gives

$$p = 10q^2 + 2q = 10 \cdot 3^2 + 2 \cdot 3 = 10 \cdot 9 + 6 = 90 + 6 = \$96$$

Graphical Method: Graph the two functions on a graphing calculator, and use the intersection finder to determine that the equilibrium point is $(3, 96)$, as in the Figure on the right.



EXAMPLE: The rental manager of a small apartment complex with 16 units has found from experience that each \$40 increase in the monthly rent results in an empty apartment. All 16 apartments will be rented at a monthly rent of \$500. How many \$40 increases will produce maximum monthly income for the complex?

Solution: Let x represent the number of \$40 increases. Then the number of apartments rented will be $16 - x$. Also, the monthly rent per apartment will be $500 + 40x$. (There are x increases of \$40, for a total increase of $40x$.) The monthly income, $I(x)$, is given by the number of apartments rented times the rent per apartment, so

$$\begin{aligned} I(x) &= (16 - x)(500 + 40x) \\ &= 8000 + 640x - 500x - 40x^2 \\ &= 8000 + 140x - 40x^2 \end{aligned}$$

Since x represents the number of \$40 increases and each \$40 increase causes one empty apartment, x must be a whole number. Because there are only 16 apartments, $0 \le x \le 16$. Since there is a small number of possibilities, the value of x that produces maximum income may be found in several ways.

Brute Force Method: Use a scientific calculator or the table feature of a graphing calculator (as in the Figure below) to evaluate $I(x)$ when $x = 1, 2, \ldots, 16$ and find the largest value.

X	Y1	
0	8000	
1	8100	
2	8120	
3	8060	
4	7920	
5	7700	
6	7400	
X=0		

X	Y1	
7	7020	
8	6560	
9	6020	
10	5400	
11	4700	
12	3920	
13	3060	
X=7		

X	Y1	
14	2120	
15	1100	
16	0	
17	-1180	
18	-2440	
19	-3780	
20	-5200	
X=14		

The tables show that a maximum income of \$8120 occurs when $x = 2$. So the manager should charge rent of $500 + 2(40) = \$580$, leaving two apartments vacant.

Algebraic Method: The graph of $I(x) = 8000 + 140x - 40x^2$ is a downward-opening parabola (since $a = -40 < 0$), and the value of x that produces maximum income occurs at the vertex. Since $a = -40$ and $b = 140$, the x-value of the vertex is

$$x = -\frac{b}{2a} = -\frac{140}{2(-40)} = 1.75$$

The y-value of the vertex is

$$I(1.75) = 8000 + 140(1.75) - 40(1.75)^2 = 8122.50$$

Therefore the vertex is (1.75, 8122.50). Since x must be a whole number, evaluate $I(x)$ at $x = 1$ and $x = 2$ to see which one gives the best result:

$$\text{If } x = 1, \text{ then } I(1) = 8000 + 140(1) - 40(1)^2 = 8100.$$
$$\text{If } x = 2, \text{ then } I(2) = 8000 + 140(2) - 40(2)^2 = 8120.$$

So maximum income occurs when $x = 2$. The manager should charge a rent of $500 + 2(40) = \$580$, leaving two apartments vacant.