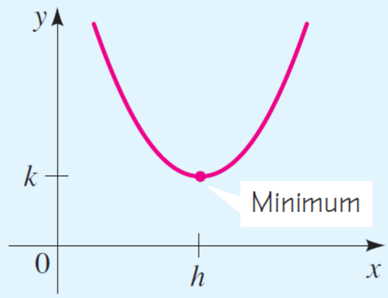
Maximum and Minimum Values of Quadratic Functions

Maximum or Minimum Value of a Quadratic Function

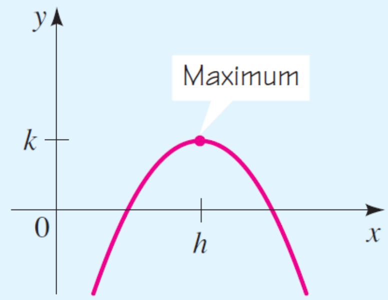
Let f be a quadratic function with standard form $f(x) = a(x-h)^2 + k$. The maximum or minimum value of f occurs at $x = h$.

If $a > 0$, then the **minimum value** of f is $f(h) = k$.

If $a < 0$, then the **maximum value** of f is $f(h) = k$.



$f(x) = a(x-h)^2 + k, a > 0$



$f(x) = a(x-h)^2 + k, a < 0$

EXAMPLE: Consider the quadratic function $f(x) = 5(x-3)^2 + 4$.

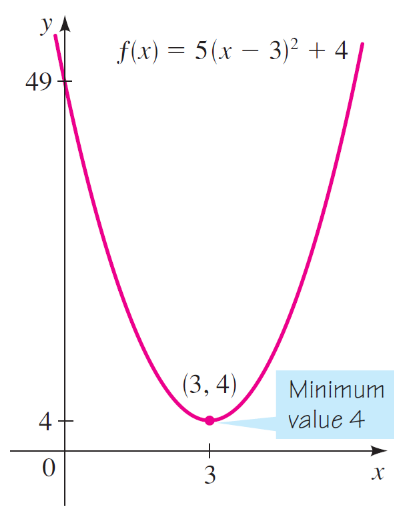
(a) Find the maximum or minimum value of f.

(b) Sketch the graph of f.

Solution:

(a) Since the coefficient of x^2 is positive, f has a *minimum* value. The minimum value is $f(3) = 4$.

(b) The graph is a parabola that has its vertex at $(3, 4)$ and opens upward, as sketched in the Figure below.

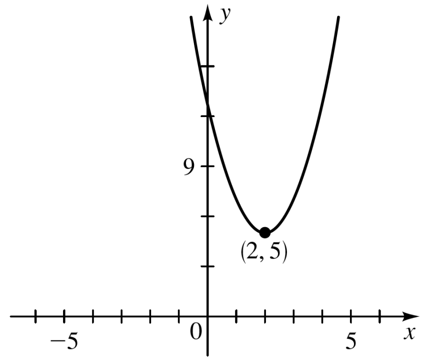


EXAMPLE: Consider the quadratic function $f(x) = 2(x-2)^2 + 5$. Find the maximum or minimum value of f. Then sketch the graph of f.

EXAMPLE: Consider the quadratic function $f(x) = 2(x-2)^2 + 5$. Find the maximum or minimum value of f. Then sketch the graph of f.

Solution: Since the coefficient of x^2 is positive, f has a *minimum* value. The minimum value is $f(2) = 5$.

The graph is a parabola that has its vertex at $(2, 5)$ and opens upward, as sketched in the Figure on the right.



EXAMPLE: Consider the quadratic function $f(x) = -\left(x - \dfrac{1}{2}\right)^2 + \dfrac{9}{4}$.

(a) Find the maximum or minimum value of f.

(b) Find the x- and y-intercepts of f.

(c) Sketch the graph of f.

Solution:

(a) Since the coefficient of x^2 is negative, f has a *maximum* value, which is $f\left(\dfrac{1}{2}\right) = \dfrac{9}{4}$.

(b) The y-intercept is $(0, 2)$, since

$$f(0) = -\left(0 - \frac{1}{2}\right)^2 + \frac{9}{4} = -\left(-\frac{1}{2}\right)^2 + \frac{9}{4} = -\frac{1}{4} + \frac{9}{4} = \frac{-1+9}{4} = \frac{8}{4} = 2$$

To find the x-intercepts, we set $f(x) = 0$:

$$-\left(x - \frac{1}{2}\right)^2 + \frac{9}{4} = 0$$

$$\frac{9}{4} = \left(x - \frac{1}{2}\right)^2$$

$$\sqrt{\frac{9}{4}} = \sqrt{\left(x - \frac{1}{2}\right)^2}$$

$$\frac{3}{2} = \left|x - \frac{1}{2}\right|$$

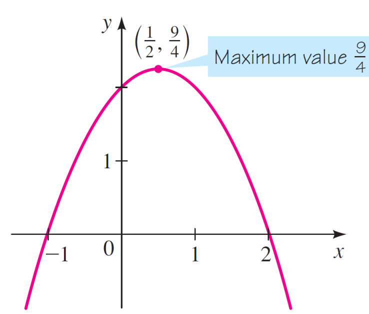
$$\pm\frac{3}{2} = x - \frac{1}{2}$$

therefore

$$x = \frac{1}{2} \pm \frac{3}{2} = \frac{1 \pm 3}{2} \quad \Longrightarrow \quad x = -1,\ 2$$

Thus, the x-intercepts are $(-1, 0)$ and $(2, 0)$.

(c) The graph of f is sketched in the Figure on the right.



Maximum or Minimum Value of a Quadratic Function

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\frac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\frac{b}{2a}\right)$.

EXAMPLE: Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$ (b) $g(x) = -2x^2 + 4x - 5$

Solution:

(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus, the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value

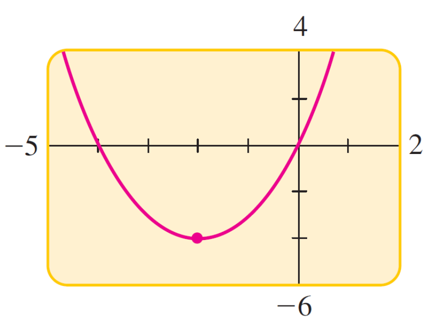
$$f(-2) = (-2)^2 + 4(-2) = -4$$

(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus, the maximum or minimum value occurs at

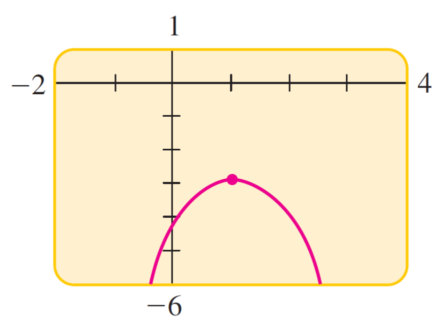
$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot (-2)} = 1$$

Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$



The minimum value occurs at $x = -2$.



The maximum value occurs at $x = 1$.