Section 3.3 Applications of Linear Functions

Most of this section deals with the following basic business relationships:

> **Revenue = (Price per item) × (Number of items)**
>
> **Cost = Fixed Costs + Variable Costs**
>
> **Profit = Revenue - Cost**

Cost Analysis

EXAMPLE: An anticlot drug can be made for $10 per unit. The total cost to produce 100 units is $1500.

(a) Assuming that the cost function is linear, find its rule.

Solution: Since the cost function $C(x)$ is linear, its rule is of the form $C(x) = mx + b$. We are given that m (the cost per item) is 10, so the rule is

$$C(x) = 10x + b$$

To find b, use the fact that it costs $1500 to produce 100 units, which means that

$$C(100) = 1500$$

$$10(100) + b = 1500$$

$$1000 + b = 1500$$

$$b = 1500 - 1000 = 500$$

So the rule of the cost function is $C(x) = 10x + 500$.

(b) What are the fixed costs?

Solution: The fixed costs are

$$C(0) = 10(0) + 500 = \$500$$

DEFINITION: If $C(x)$ is the total cost to produce x items, then the **average cost** per item is given by

$$\boxed{\overline{C}(x) = \frac{C(x)}{x}}$$

REMARK: As more and more items are produced, the average cost per item typically decreases.

EXAMPLE: Find the average cost of producing 100 and 1000 units of the anticlot drug in the previous Example.

Solution: The cost function is $C(x) = 10x + 500$, so the average cost of producing 100 units is

$$\overline{C}(100) = \frac{C(100)}{100} = \frac{10(100) + 500}{100} = \frac{1000 + 500}{100} = \frac{1500}{100} = \$15.00 \text{ per unit}$$

The average cost of producing 1000 units is

$$\overline{C}(1000) = \frac{C(1000)}{1000} = \frac{10(1000) + 500}{1000} = \frac{10,000 + 500}{1000} = \frac{10,500}{1000} = \$10.50 \text{ per unit}$$

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Rates of Change

> **The rate of change of a linear function $f(x) = mx + b$ is the slope m**

The value of a computer, or an automobile, or a machine *depreciates* (decreases) over time. **Linear depreciation** means that the value of the item at time x is given by a linear function $f(x) = mx + b$. The slope m of this line gives the rate of depreciation.

EXAMPLE: According to the *Kelley Blue Book*, a Ford Mustang two-door convertible that is worth \$14,776 today will be worth \$10,600 in two years (if it is in excellent condition with average mileage).

(a) Assuming linear depreciation, find the depreciation function for this car.

Solution: We know the car is worth \$14,776 now $(x = 0)$ and will be worth \$10,600 in two years $(x = 2)$. So the points $(0, 14{,}776)$ and $(2, 10{,}600)$ are on the graph of the linear depreciation function and can be used to determine its slope:

$$m = \frac{10,600 - 14,776}{2} = \frac{-4176}{2} = -2088$$

Using the point (0, 14,776), we find that the equation of the line is

$$y - 14,776 = -2088(x - 0)$$

$$y - 14,776 = -2088x$$

$$y = -2088x + 14,776.$$

Therefore, the rule of the depreciation function is $f(x) = -2088x + 14,776$.

(b) What will the car be worth in 4 years?

Solution: Evaluate f when $x = 4$:

$$f(x) = -2088x + 14,776$$

$$f(4) = -2088(4) + 14,776 = \$6424$$

(c) At what rate is the car depreciating?

Solution: The depreciation rate is given by the slope of $f(x) = -2088x + 14,776$, namely, -2088. This negative slope means that the car is decreasing in value an average of \$2088 a year.

DEFINITION: The rate of change of the cost function is called the **marginal cost**.

REMARK 1: Marginal cost is important to management in making decisions in such areas as cost control, pricing, and production planning. When the cost function is linear, say, $C(x) = mx + b$, the marginal cost is m (the *slope* of the graph of C).

REMARK 2: Marginal cost can also be thought of as the *cost of producing one more item.*

EXAMPLE: An electronics company manufactures handheld PCs. The cost function for one of its models is $C(x) = 160x + 750,000$.

(a) What are the fixed costs for this product?

Solution: The fixed costs are

$$C(0) = 160(0) + 750,000 = \$750,000$$

(b) What is the marginal cost?

Solution: The slope of $C(x) = 160x + 750,000$ is 160, so the marginal cost is \$160 per item.

(c) After 50,000 units have been produced, what is the cost of producing one more?

Solution: We have

$C(50,000) = 160(50,000) + 750,000 = \$8,750,000$ (cost of producing 50,000 units)

$C(50,001) = 160(50,001) + 750,000 = \$8,750,160$ (cost of producing 50,001 units)

The cost of the additional unit is the difference

$$C(50,001) - C(50,000) = 8,750,160 - 8,750,000 = \$160$$

Thus, the cost of one more item, is the marginal cost.

DEFINITION: The rate of change of a revenue function is called the **marginal revenue**.

REMARK: When the revenue function is linear, the marginal revenue is the *slope* of the line, as well as the *revenue from producing one more item.*

EXAMPLE: The energy company New York State Electric and Gas charges each residential customer a basic fee for electricity of \$15.11, plus \$.0333 per kilowatt hour (kWh).

(a) Assuming there are 700,000 residential customers, find the company's revenue function.

(b) What is the marginal revenue?

EXAMPLE: The energy company New York State Electric and Gas charges each residential customer a basic fee for electricity of $15.11, plus $.0333 per kilowatt hour (kWh).

(a) Assuming there are 700,000 residential customers, find the company's revenue function.

Solution: The monthly revenue from the basic fee is

$$15.11(700,000) = \$10,577,000$$

If x is the total number of kilowatt hours used by all customers, then the revenue from electricity use is .0333x. So the monthly revenue function is given by

$$R(x) = .0333x + 10,577,000$$

(b) What is the marginal revenue?

Solution: The marginal revenue (the rate at which revenue is changing) is given by the slope of the rate function: $0.0333 per kWh.

The last two Examples are typical of the general case, as summarized here.

In a **linear cost function** $C(x) = mx + b$, the marginal cost is m (the slope of the cost line) and the fixed cost is b (the y-intercept of the cost line). The marginal cost is the cost of producing one more item.

Similarly, in a **linear revenue function** $R(x) = kx + d$, the marginal revenue is k (the slope of the revenue line), which is the revenue from selling one more item.

Break-Even Analysis

A typical company must analyze its costs and the potential market for its product to determine when (or even whether) it will make a profit.

EXAMPLE: A company manufactures a 42-inch plasma HDTV that sells to retailers for $550. The cost of making x of these TVs for a month is given by the cost function $C(x) = 250x + 213,000$.

(a) Find the function R that gives the revenue from selling x TVs.

Solution: Since revenue is the product of the price per item and the number of items, $R(x) = 550x$.

(b) What is the revenue from selling 600 TVs?

Solution: Evaluate the revenue function R at 600:

$$R(600) = 550(600) = \$330,000$$

(c) Find the profit function P. Then find the profit from selling 500 TVs.

Solution: Since Profit = Revenue − Cost, we have

$$P(x) = R(x) - C(x) = 550x - (250x + 213,000) = 550x - 250x - 213,000 = 300x - 213,000$$

therefore

$$P(500) = 300(500) - 213,000 = 150,000 - 213,000 = -63,000$$

that is, a loss of $63,000.

REMARK: A company can make a profit only if the revenue on a product exceeds the cost of manufacturing it.

DEFINITION: The number of units at which revenue equals cost (that is, profit is 0) is the **break-even point**.

EXAMPLE: Find the break-even point for the company in the previous Example.

Solution: The company will break even when revenue equals cost — that is, when
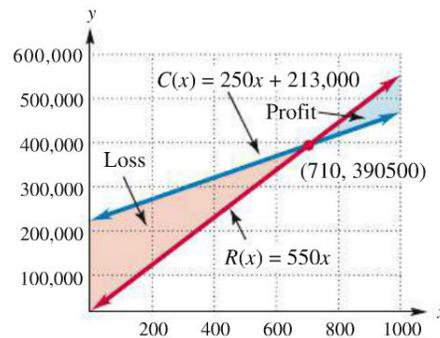
$$R(x) = C(x)$$

$$550x = 250x + 213,000$$

$$300x = 213,000$$

$$x = \frac{213,000}{300} = 710$$

The company breaks even by selling 710 TVs. The graphs of the revenue and cost functions and the break-even point (where $x = 710$) are shown in the Figure below. The company must sell more than 710 TVs ($x > 710$) in order to make a profit.

Supply and Demand

The supply of and demand for an item are usually related to its price. Producers will supply large numbers of the item at a high price, but consumer demand will be low. As the price of the item decreases, consumer demand increases, but producers are less willing to supply large numbers of the item.

DEFINITION: The curves showing the quantity that will be supplied at a given price and the quantity that will be demanded at a given price are called **supply and demand curves**, respectively.

EXAMPLE: Joseph Nolan has studied the supply and demand for aluminum siding and has determined that the price per unit, p, and the quantity demanded, q, are related by the linear equation

$$p = 60 - \frac{3}{4}q$$

(a) Find the demand at a price of $40 per unit.

Solution: Let $p = 40$. Then we have

$$p = 60 - \frac{3}{4}q$$

$$40 = 60 - \frac{3}{4}q$$

$$40 - 60 = 60 - \frac{3}{4}q - 60$$

$$-20 = -\frac{3}{4}q$$

$$-20 \cdot \left(-\frac{4}{3}\right) = -\frac{3}{4}q \cdot \left(-\frac{4}{3}\right)$$

$$\frac{80}{3} = q$$

At a price of $40 per unit, $80/3$ (or $26\frac{2}{3}$) units will be demanded.

(b) Find the price if the demand is 32 units.

Solution: Let $q = 32$. Then we have

$$p = 60 - \frac{3}{4}q$$

$$p = 60 - \frac{3}{4}(32)$$

$$p = 60 - 3(8)$$

$$p = 60 - 24$$

$$p = 36$$

With a demand of 32 units, the price is $36.

(c) Graph $p = 60 - \dfrac{3}{4}q$.

Solution: To graph $p = 60 - \dfrac{3}{4}q$, we can plug any two numbers in for x. For example, if $q = 0$, then

$$p = 60 - \frac{3}{4}q = 60 - \frac{3}{4}(0) = 60$$

Similarly, if $q = 32$, then $p = 36$ by (b). Using the points $(0, 60)$ and $(32, 36)$ yields the demand graph depicted in the Figure on the right. Only the portion of the graph in Quadrant I is shown, because supply and demand are meaningful only for positive values of p and q.
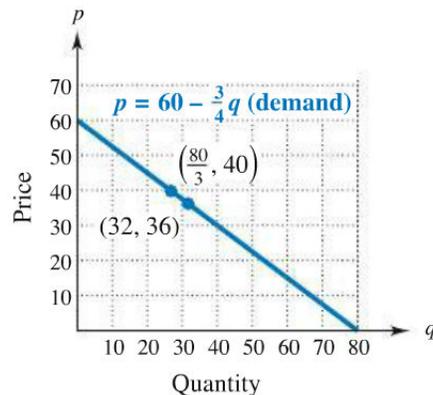
(d) From the Figure above, at a price of \$30, what quantity is demanded?

Solution: Price is located on the vertical axis. Look for 30 on the p-axis, and read across to where the line $p = 30$ crosses the demand graph. As the graph shows, this occurs where the quantity demanded is 40.

(e) At what price will 60 units be demanded?

Solution: Quantity is located on the horizontal axis. Find 60 on the q-axis, and read up to where the vertical line $q = 60$ crosses the demand graph. This occurs where the price is about \$15 per unit.

(f) What quantity is demanded at a price of \$60 per unit?

Solution: The point $(0, 60)$ on the demand graph shows that the demand is 0 at a price of \$60 (that is, there is no demand at such a high price).

EXAMPLE: Suppose the economist in the previous Example concludes that the supply q of siding is related to its price p by the equation

$$p = .85q$$

(a) Find the supply if the price is \$51 per unit.

Solution: We have

$$51 = .85q \quad \Longrightarrow \quad q = \frac{51}{.85} = 60$$

If the price is \$51 per unit, then 60 units will be supplied to the marketplace.

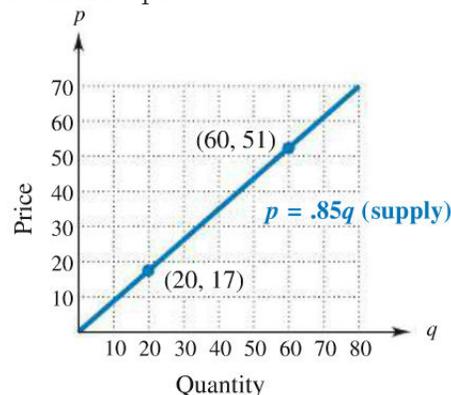(b) Find the price per unit if the supply is 20 units.

Solution: We have

$$p = .85(20) = 17$$

If the supply is 20 units, then the price is \$17 per unit.

(c) Graph the supply equation $p = .85q$.

Solution: Part (a) shows that the ordered pair $(60, 51)$ is on the graph of the supply equation, and part (b) shows that $(20, 17)$ is on the graph.

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EXAMPLE: The supply and demand curves of the last two Examples are shown in the Figure below. Determine graphically whether there is a surplus or a shortage of supply at a price of $40 per unit.
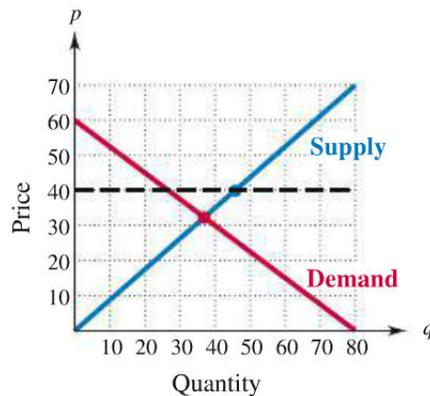
Solution: Find 40 on the vertical axis in the Figure above and read across to the point where the horizontal line $p = 40$ crosses the supply graph (that is, the point corresponding to a price of $40). This point lies above the demand graph, so supply is greater than demand at a price of $40, and there is a surplus of supply.

DEFINITION: The **equilibrium point** is the point where supply and demand are equal, that is, where the supply curve intersects the demand curve. Its second coordinate is the **equilibrium price**, the price at which the same quantity will be supplied as is demanded. Its first coordinate is the quantity that will be demanded and supplied at the equilibrium price; this number is called the **equilibrium quantity**.

EXAMPLE: In the situation described in the last three Examples, what is the equilibrium quantity? What is the equilibrium price?

Solution: The equilibrium point is where the supply and demand curves in the Figure above intersect. To find the quantity q at which the price given by the demand equation $p = 60 - .75q$ is the same as that given by the supply equation $p = .85q$, set these two expressions for p equal to each other and solve the resulting equation:

$$60 - .75q = .85q$$

$$60 = .85q + .75q$$

$$60 = 1.6q$$

$$q = \frac{60}{1.6} = 37.5$$

Therefore, the equilibrium quantity is 37.5 units, the number of units for which supply will equal demand. Substituting $q = 37.5$ into either the demand or supply equation shows that

$$p = 60 - .75(37.5) = 31.875 \quad \text{or} \quad p = .85(37.5) = 31.875$$

to substitute into both equations, as we did here, to be sure that the same value of p results; if it does not, a mistake has been made.) In this case, the equilibrium point-the point whose coordinates are the equilibrium quantity and price — is $(37.5, 31.875)$.