Section 3.2 Graphs of Functions

DEFINITION: A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. It's **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

EXAMPLE:

Sketch the graphs of the following functions.

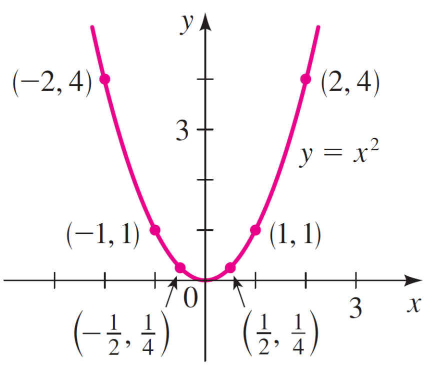
(a) $f(x) = x^2$ (b) $g(x) = x^3$ (c) $h(x) = \sqrt{x}$

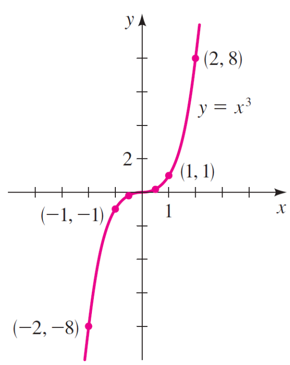
Solution: We first make a table of values. Then we plot the points given by the table and join them by a smooth curve to obtain the graph. The graphs are sketched in the Figures below.

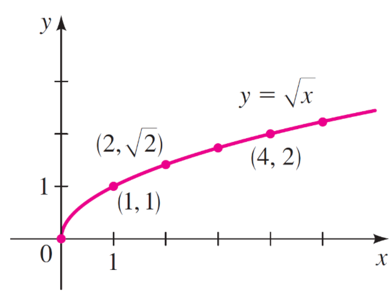
x	$f(x) = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

x	$g(x) = x^3$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
$-\frac{1}{2}$	$-\frac{1}{8}$
-1	-1
-2	-8

x	$h(x) = \sqrt{x}$
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{3}$
4	2
5	$\sqrt{5}$







EXAMPLE: Sketch the graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

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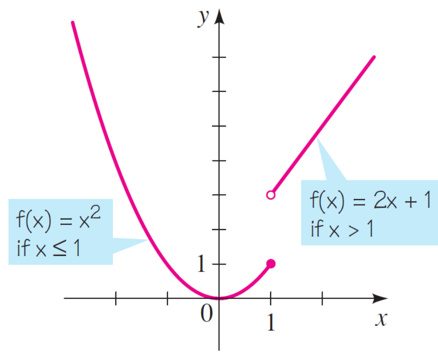
$$f(x)=\begin{cases} x^2 & \text{if } x\le 1 \\ 2x+1 & \text{if } x>1 \end{cases}$$

Solution: If $x\le 1$, then $f(x)=x^2$, so the part of the graph to the left of $x=1$ coincides with the graph of $y=x^2$. If $x>1$, then $f(x)=2x+1$, so the part of the graph to the right of $x=1$ coincides with the line $y=2x+1$.

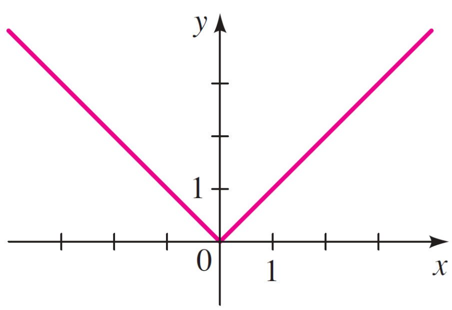


EXAMPLE: Sketch the graph of the function $f(x)=|x|$.

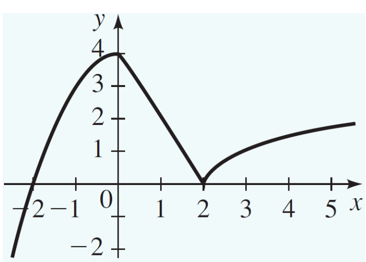
Solution: Recall that

$$f(x)=|x|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$$

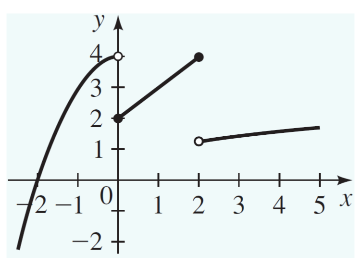
Using the same method as in the previous example, we note that the graph of f coincides with the line $y=x$ to the right of the y-axis and coincides with the line $y=-x$ to the left of the y-axis.



EXAMPLES:



$$f(x)=\begin{cases} 4-x^2 & \text{if } x<0 \\ 4-2x & \text{if } 0\le x\le 2 \\ \sqrt{x-2} & \text{if } x>2 \end{cases}$$



$$f(x)=\begin{cases} 4-x^2 & \text{if } x<0 \\ x+2 & \text{if } 0\le x\le 2 \\ \sqrt[3]{x} & \text{if } x>2 \end{cases}$$

Step Functions

The **greatest-integer function**, usually written $f(x) = [x]$, is defined by saying that $[x]$ denotes the largest integer that is less than or equal to x . For example,

$$[8] = 8, \quad [7.45] = 7, \quad [8.99] = 8, \quad [\pi] = 3, \quad [-1] = -1, \quad [-2.6] = -3, \quad [-5.1] = -6$$

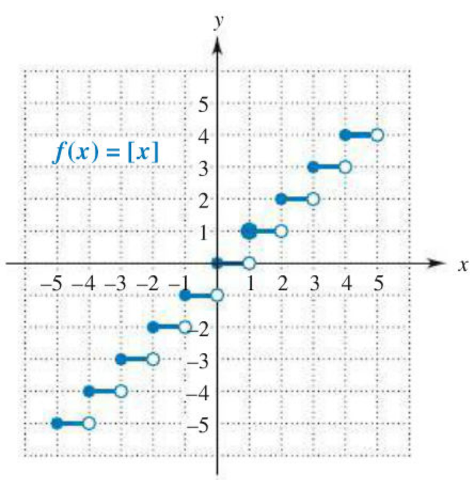
and so on.

EXAMPLE: Graph the greatest-integer function $f(x) = [x]$.

Solution: Consider the values of the function between each two consecutive integers — for instance,

x	$-2 \leq x < -1$	$-1 \leq x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$
$[x]$	-2	-1	0	1	2

Thus, between $x = -2$ and $x = -1$, the value of $f(x) = [x]$ is always -2, so the graph there is a horizontal line segment, all of whose points have second coordinate -2. The rest of the graph is obtained similarly (see the Figure below). An open circle in that figure indicates that the endpoint of the segment is *not* on the graph, whereas a closed circle indicates that the endpoint *is* on the graph.



Functions whose graphs resemble the graph of the greatest-integer function are sometimes called **step functions**.

EXAMPLE: In 2013, the U.S. Post Office charged to ship a flat envelope first class to Eastern Europe, Europe, or Australia a fee of \$2.05 for up to and including the first ounce, \$.85 for each additional ounce or fraction of an ounce up to and including 8 ounces, and then \$1.70 for each additional four ounces or less, up to a peak of 64 ounces. Let $D(x)$ represent the cost to send a flat envelope weighing x ounces. Graph $D(x)$ for x in the interval $(0, 20]$.

Solution:

For x in the interval $(0, 1]$, $y = 2.05$.

For x in $(1, 2]$, $y = 2.05 + .85 = 2.90$.

For x in $(2, 3]$, $y = 2.90 + .85 = 3.75$.

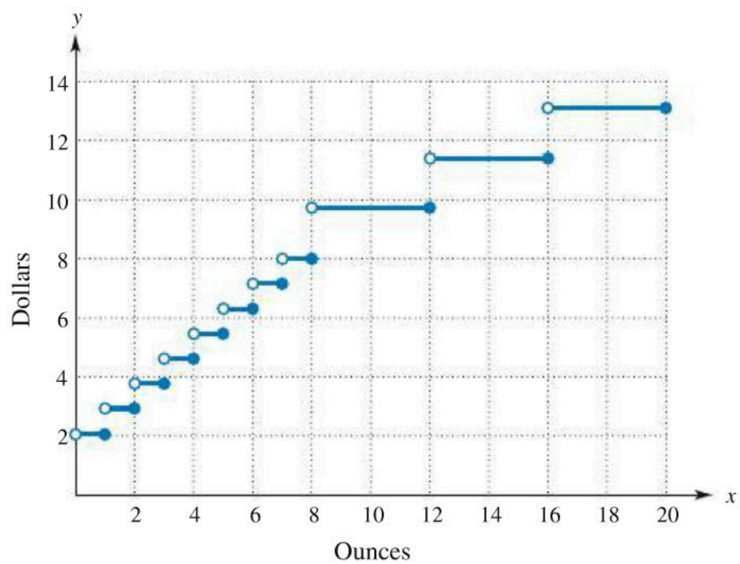
And so on up to x in $(7, 8]$, $y = 7.15 + .85 = 8.00$.

Then for x in $(8, 12]$, $y = 8.00 + 1.70 = 9.70$.

For x in $(12, 16]$, $y = 9.70 + 1.70 = 11.40$.

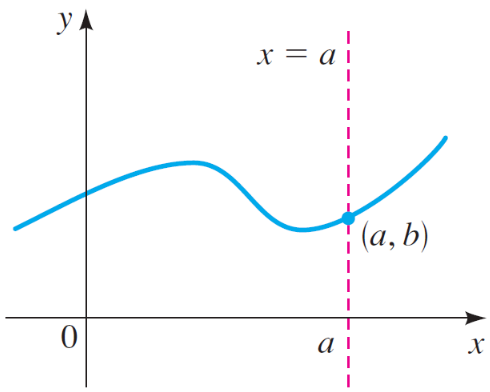
For x in $(16, 20]$, $y = 11.40 + 1.70 = 13.10$.

The graph, which is that of a step function, is shown in the Figure below.

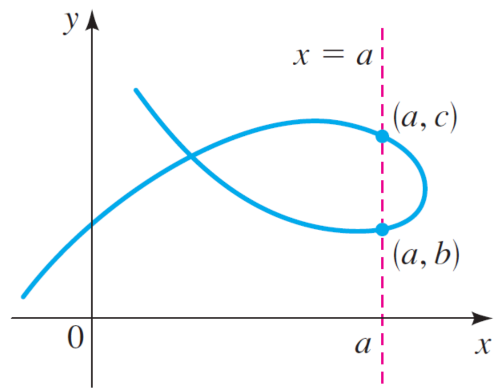


The Vertical Line Test

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

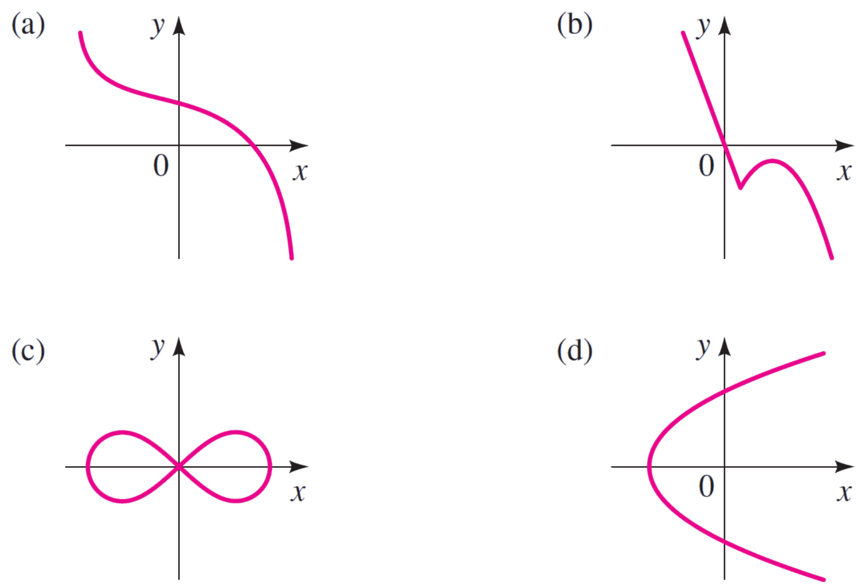


Graph of a function



Not a graph of a function

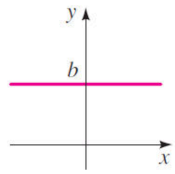
EXAMPLE: Which of the following are graphs of functions?



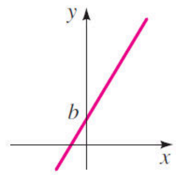
Solution: (a) and (b) are graphs of functions, (c) and (d) are not.

Some Functions and Their Graphs

Linear functions
$f(x) = mx + b$

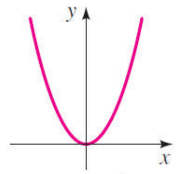


$f(x) = b$

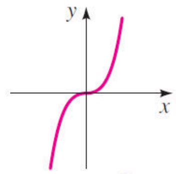


$f(x) = mx + b$

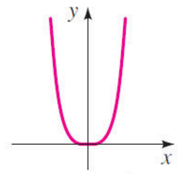
Power functions
$f(x) = x^n$



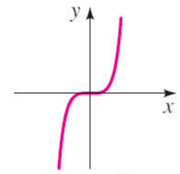
$f(x) = x^2$



$f(x) = x^3$

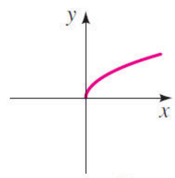


$f(x) = x^4$

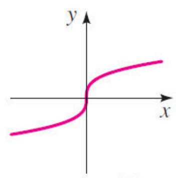


$f(x) = x^5$

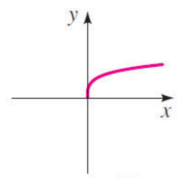
Root functions
$f(x) = \sqrt[n]{x}$



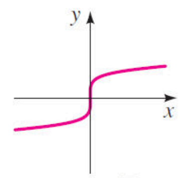
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$

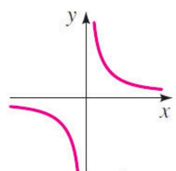


$f(x) = \sqrt[4]{x}$

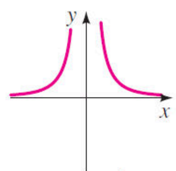


$f(x) = \sqrt[5]{x}$

Reciprocal functions
$f(x) = 1/x^n$

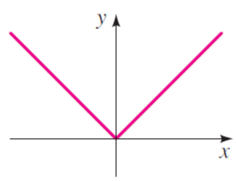


$f(x) = \dfrac{1}{x}$



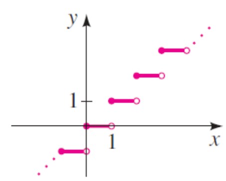
$f(x) = \dfrac{1}{x^2}$

Absolute value function
$f(x) = |x|$



$f(x) = |x|$

Greatest integer function
$f(x) = [\![x]\!]$



$f(x) = [\![x]\!]$