Section 3.1 Functions

DEFINITION: A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of f. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

REMARK: In a function, each input produces a single output, but different inputs may produce the same output.

EXAMPLES: Which of the following rules describe functions?

(a) Use the optical reader at the checkout counter of the supermarket to convert codes to prices.

Solution: For each code, the reader produces exactly one price, so this is a function.

(b) The correspondence between a computer, x, and several users of the computer, y.

Solution: Since for a computer x there are several users y, this correspondence is not a function.

(c) Assign to each number x the number y given by this table:

x	1	1	2	2	3	3
y	3	-3	-5	-5	8	-8

Solution: Since $x=1$ corresponds to more than one y-value (as does $x=3$), this table does not define a function.

(d) Assign to each number x the number y given by the equation $y=3x-5$.

Solution: Because the equation determines a unique value of y for each value of x, it defines a function.

EXAMPLES: Decide whether each of the following equations defines y as a function of x.

(a) $y=x^2$

Solution: For a given value of x, calculating x^2 produces exactly one value of y. Because one value of the input variable leads to exactly one value of the output variable, $y=x^2$ defines y as a function of x.

(b) $x=y^2$

Solution: Suppose $x=1$. Then $y^2=x$ becomes $y^2=1$, from which $y=1$ or $y=-1$. Since one value of x can lead to two values of y, $y^2=x$ does not define y as a function of x.

(c) $x=y^3$

Solution: We have

$$x=y^3 \iff \sqrt[3]{x}=\sqrt[3]{y^3} \iff \sqrt[3]{x}=y$$

For a given value of x, calculating $\sqrt[3]{x}$ produces exactly one value of y. Because one value of the input variable leads to exactly one value of the output variable, $\sqrt[3]{x}=y$ (and therefore $x=y^3$) defines y as a function of x.

EXAMPLES:

1. Let $f(x)=x+\sqrt{x}$. Then

$$f(0)=0+\sqrt{0}=0$$
$$f(4)=4+\sqrt{4}=6$$

2. Let

$$f(x)=\begin{cases} 3x^2+x-5 & \text{if } x\le 0 \\ x+\sqrt{x} & \text{if } x>0 \end{cases} \qquad \Longrightarrow \qquad \begin{aligned} f(-2)&=3\cdot(-2)^2+(-2)-5=5 \\ f(0)&=3\cdot 0^2+0-5=-5 \\ f(4)&=4+\sqrt{4}=6 \end{aligned}$$

3. Let $f(x)=\dfrac{\sqrt{x+1}}{x}$. Then

$$f(a-1)=\frac{\sqrt{a-1+1}}{a-1}=\frac{\sqrt{a}}{a-1}$$

REMARK: Note that

$$f(a-1)\neq\frac{\sqrt{a+1}}{a}-1$$

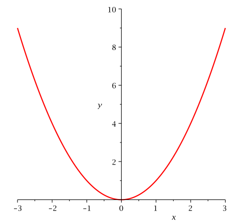
4. Let $f(x)=-x^2+4x-5$. Then

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\Big(-(x+h)^2+4(x+h)-5\Big)-\Big(-x^2+4x-5\Big)}{h}\\ &=\frac{\Big(-x^2-2xh-h^2+4x+4h-5\Big)-\Big(-x^2+4x-5\Big)}{h}\\ &=\frac{-x^2-2xh-h^2+4x+4h-5+x^2-4x+5}{h}\\ &=\frac{-2xh-h^2+4h}{h}=\frac{h(-2x-h+4)}{h}=-2x-h+4\end{aligned}$$

REMARK: $\dfrac{f(x+h)-f(x)}{h}$ is called the **difference quotient** of the function f. Difference quotients are important in calculus.

Domain (*and Range*)

EXAMPLES:

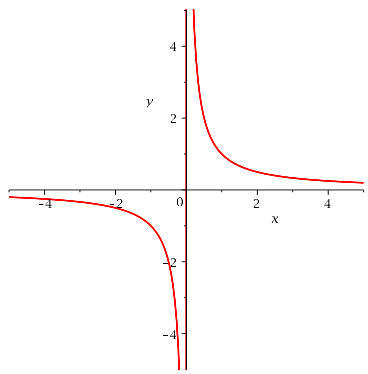


1. $f(x) = x^2$

Domain: All real numbers or $(-\infty, \infty)$. Range: $[0, \infty)$.

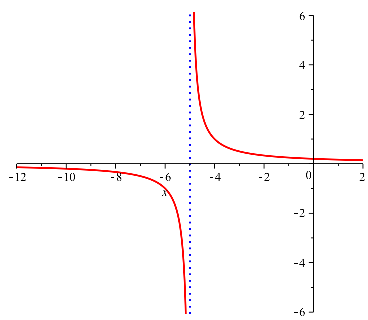
2. $f(x) = \dfrac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$, since $x \neq 0$. Range: $(-\infty, 0) \cup (0, \infty)$.

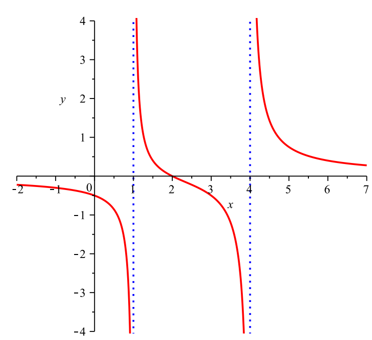


3. $f(x) = \dfrac{1}{x+5}$

Domain: $(-\infty, -5) \cup (-5, \infty)$, since $x + 5 \neq 0$.



4. $f(x) = \dfrac{x-2}{x^2 - 5x + 4}$



Domain: All real numbers except 1 and 4:

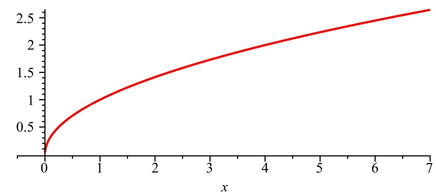
$$(-\infty, 1) \cup (1, 4) \cup (4, \infty)$$

since

$$x^2 - 5x + 4 = (x-1)(x-4) \neq 0$$

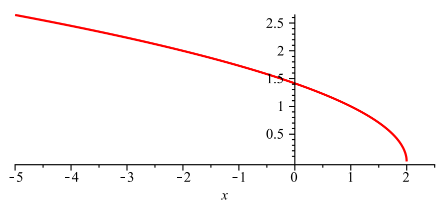
5. $f(x) = \sqrt{x}$

Domain: $[0, \infty)$.
Range: $[0, \infty)$.



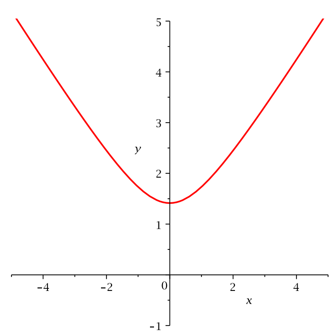
6. $f(x) = \sqrt{2 - x}$

Domain: $(-\infty, 2]$, since $2 - x$



7. $f(x) = \sqrt{x^2 + 2}$

Domain: All real numbers, or $(-\infty, \infty)$, since $x^2 + 2$ is always > 0.



Applications

EXAMPLE: If you were a single person in Connecticut in 2013 with a taxable income of x dollars and $x \leq \$500,000$ then your state income tax T was determined by the rule

$$T(x) = \begin{cases} .03x & \text{if } 0 \leq x \leq 10,000 \\ \\ 300 + .05(x - 10,000) & \text{if } 10,000 \leq x \leq 500,000 \end{cases}$$

Find the income tax paid by a single person with the given taxable income. (Data from: www.tax.brackets.org.)

(a) \$9200

Solution: We must find $T(9200)$. Since 9200 is less than 10,000, the first part of the rule applies:

$$T(x) = .03x$$

$$T(9200) = .03(9200) = \$276$$

(b) \$30,000

Solution: Now we must find $T(30,000)$. Since 30,000 is greater than \$10,000, the second part of the rule applies:

$$\begin{aligned} T(x) &= 300 + .05(x - 10,000) \\ T(30,000) &= 300 + .05(30,000 - 10,000) \\ &= 300 + .05(20,000) \\ &= 300 + 1000 \\ &= \$1300 \end{aligned}$$