Section 3.1 Functions

DEFINITION: A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of f. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

REMARK: In a function, each input produces a single output, but different inputs may produce the same output.

EXAMPLES: Which of the following rules describe functions?

(a) Use the optical reader at the checkout counter of the supermarket to convert codes to prices.

Solution: For each code, the reader produces exactly one price, so this is a function.

(b) The correspondence between a computer, x, and several users of the computer, y.

Solution: Since for a computer x there are several users y, this correspondence is not a function.

(c) Enter a number in a calculator and press the x^2 key.

Solution: This is a function, because the calculator produces just one number x^2 for each number x that is entered.

(d) Assign to each number x the number y given by this table:

x	1	1	2	2	3	3
y	3	-3	-5	-5	8	-8

Solution: Since $x = 1$ corresponds to more than one y-value (as does $x = 3$), this table does not define a function.

(e) Assign to each number x the number y given by the equation $y = 3x - 5$.

Solution: Because the equation determines a unique value of y for each value of x, it defines a function.

EXAMPLES: Decide whether each of the following equations defines y as a function of x.

(a) $y = -4x + 11$

Solution: For a given value of x, calculating $-4x + 11$ produces exactly one value of y. (For example, if $x = -7$, then $y = -4(-7) + 11 = 39$). Because one value of the input variable leads to exactly one value of the output variable, $y = -4x + 11$ defines y as a function of x.

(b) $y = x^2$

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Solution: For a given value of x, calculating x^2 produces exactly one value of y. Because one value of the input variable leads to exactly one value of the output variable, $y = x^2$ defines y as a function of x.

(c) $x = y^2$

Solution 1: Suppose $x = 1$. Then $y^2 = x$ becomes $y^2 = 1$, from which $y = 1$ or $y = -1$. Since one value of x can lead to two values of y, $y^2 = x$ does not define y as a function of x.

Solution 2: We have

$$x = y^2 \quad \Longleftrightarrow \quad \sqrt{x} = \sqrt{y^2} \quad \Longleftrightarrow \quad \sqrt{x} = |y| \quad \text{(see the Appendix)}$$

Suppose $x = 1$. Then $\sqrt{x} = |y|$ becomes

$$\sqrt{1} = |y| \quad \Longleftrightarrow \quad 1 = |y|$$

from which $y = 1$ or $y = -1$. Since one value of x can lead to two values of y, $\sqrt{x} = |y|$ (and therefore $x = y^2$) does not define y as a function of x.

(d) $x = y^3$

Solution: We have

$$x = y^3 \quad \Longleftrightarrow \quad \sqrt[3]{x} = \sqrt[3]{y^3} \quad \Longleftrightarrow \quad \sqrt[3]{x} = y \quad \text{(see the Appendix)}$$

For a given value of x, calculating $\sqrt[3]{x}$ produces exactly one value of y. Because one value of the input variable leads to exactly one value of the output variable, $\sqrt[3]{x} = y$ (and therefore $x = y^3$) defines y as a function of x.

EXAMPLES:

1. Let $f(x) = x + \sqrt{x}$. Then

$$f(0) = 0 + \sqrt{0} = 0$$
$$f(4) = 4 + \sqrt{4} = 6$$

2. Let $f(x) = 3x^2 + x - 5$. Then

$$f(-2) = 3 \cdot (-2)^2 + (-2) - 5 = 5$$
$$f(0) = 3 \cdot 0^2 + 0 - 5 = -5$$
$$f(4) = 3 \cdot 4^2 + 4 - 5 = 47$$
$$f\left(\frac{1}{2}\right) = 3 \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} - 5 = -\frac{15}{4}$$

3. Let

$$f(x)=\begin{cases}3x^2+x-5 & \text{if } x\le 0\\ x+\sqrt{x} & \text{if } x>0\end{cases}\qquad\Longrightarrow\qquad \begin{aligned}f(-2)&=3\cdot(-2)^2+(-2)-5=5\\ f(0)&=3\cdot 0^2+0-5=-5\\ f(4)&=4+\sqrt{4}=6\end{aligned}$$

4. Let

$$g(x)=\begin{cases}3x^2+x-5 & \text{if } x< 0\\ x+\sqrt{x} & \text{if } x\ge 0\end{cases}\qquad\Longrightarrow\qquad \begin{aligned}g(-2)&=3\cdot(-2)^2+(-2)-5=5\\ g(0)&=0+\sqrt{0}=0\\ g(4)&=4+\sqrt{4}=6\end{aligned}$$

5. Let $f(x)=\dfrac{\sqrt{x+1}}{x}$. Then

$$f(3)=\frac{\sqrt{3+1}}{3}=\frac{\sqrt{4}}{3}=\frac{2}{3}\qquad\qquad f(5)=\frac{\sqrt{5+1}}{5}=\frac{\sqrt{6}}{5}$$

and

$$f(a-1)=\frac{\sqrt{a-1+1}}{a-1}=\frac{\sqrt{a}}{a-1}$$

REMARK: Note that

$$f(a-1)\neq\frac{\sqrt{a+1}}{a}-1$$

6. Let $f(x)=2x$. Then

$$\frac{f(x+h)-f(x)}{h}=\frac{2(x+h)-2x}{h}=\frac{2x+2h-2x}{h}=\frac{2h}{h}=2$$

7. Let $f(x)=-5x+7$. Then

$$\frac{f(x+h)-f(x)}{h}=\frac{\Big(-5(x+h)+7\Big)-\Big(-5x+7\Big)}{h}=\frac{-5x-5h+7+5x-7}{h}=\frac{-5h}{h}=-5$$

8. Let $f(x)=-x^2+4x-5$. Then

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\Big(-(x+h)^2+4(x+h)-5\Big)-\Big(-x^2+4x-5\Big)}{h}\\ &=\frac{\Big(-x^2-2xh-h^2+4x+4h-5\Big)-\Big(-x^2+4x-5\Big)}{h}\\ &=\frac{-x^2-2xh-h^2+4x+4h-5+x^2-4x+5}{h}\\ &=\frac{-2xh-h^2+4h}{h}=\frac{h(-2x-h+4)}{h}=-2x-h+4\end{aligned}$$

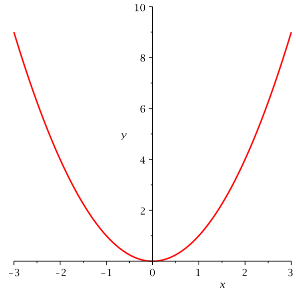
REMARK: $\dfrac{f(x+h)-f(x)}{h}$ is called the **difference quotient** of the function f. Difference quotients are important in calculus.

Domain and Range

EXAMPLES:

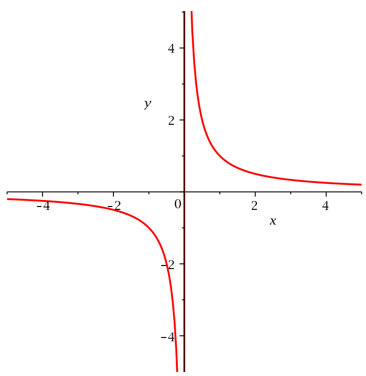
1. $f(x) = x^2$

Domain: All real numbers or $(-\infty, \infty)$.
Range: $[0, \infty)$.



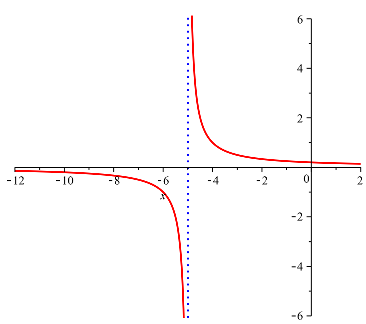
2. $f(x) = \dfrac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$, since $x \neq 0$.
Range: $(-\infty, 0) \cup (0, \infty)$.



3. $f(x) = \dfrac{1}{x+5}$

Domain: $(-\infty, -5) \cup (-5, \infty)$, since $x + 5 \neq 0$.
Range: $(-\infty, 0) \cup (0, \infty)$.



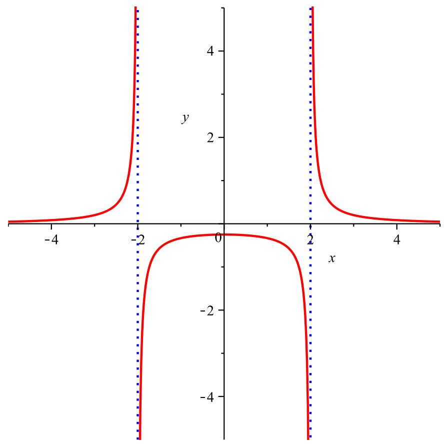
4. $f(x) = \dfrac{1}{x^2 - 4}$

Domain: All real numbers except -2 and 2:

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$$

since

$$x^2 - 4 = x^2 - 2^2 = (x-2)(x+2) \neq 0$$



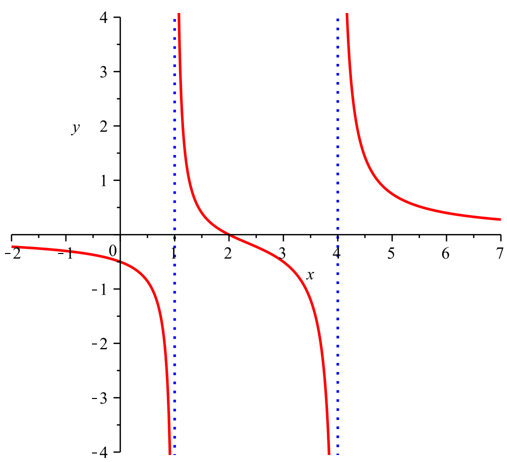
5. $f(x) = \dfrac{x - 2}{x^2 - 5x + 4}$

Domain: All real numbers except 1 and 4:

$$(-\infty, 1) \cup (1, 4) \cup (4, \infty)$$

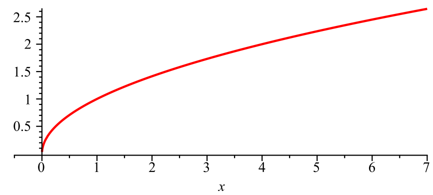
since

$$x^2 - 5x + 4 = (x-1)(x-4) \neq 0$$



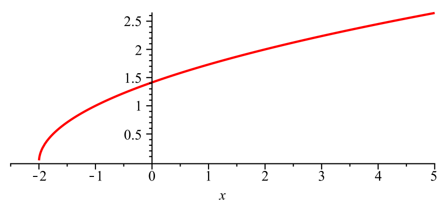
6. $f(x) = \sqrt{x}$

Domain: $[0, \infty)$.
Range: $[0, \infty)$.



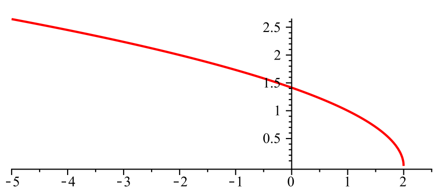
7. $f(x) = \sqrt{2 + x}$

Domain: $[-2, \infty)$, since $2 + x \geq 0$.
Range: $[0, \infty)$.



8. $f(x) = \sqrt{2 - x}$

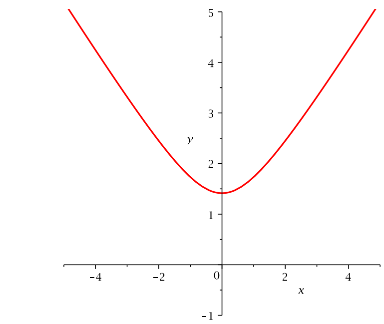
Domain: $(-\infty, 2]$, since $2 - x \geq 0$.
Range: $[0, \infty)$.



9. $f(x) = \sqrt{x^2 + 2}$

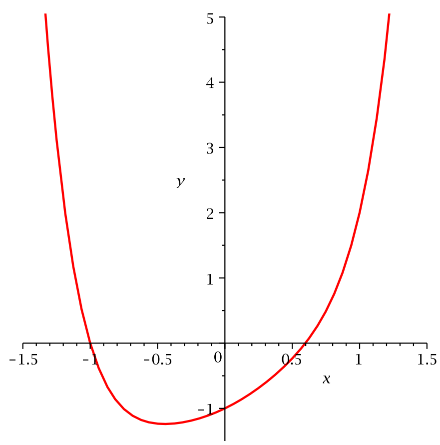
9. $f(x) = \sqrt{x^2 + 2}$

Domain: All real numbers, or $(-\infty, \infty)$, since $x^2 + 2$ is always > 0.
Range: $\{y \mid y \geq \sqrt{2}\}$ or $[\sqrt{2}, \infty)$.



10. $f(x) = x^6 + x^2 + x - 1$

Domain: $(-\infty, \infty)$.
Range: $\{y \mid y \geq -1.2392\}$ or $[-1.2392, \infty)$.



Applications

EXAMPLE: If you were a single person in Connecticut in 2013 with a taxable income of x dollars and $x \leq \$500,000$ then your state income tax T was determined by the rule

$$T(x) = \begin{cases} .03x & \text{if } 0 \leq x \leq 10,000 \\ \\ 300 + .05(x - 10,000) & \text{if } 10,000 \leq x \leq 500,000 \end{cases}$$

Find the income tax paid by a single person with the given taxable income. (Data from: www.tax.brackets.org.)

(a) \$9200

Solution: We must find $T(9200)$. Since 9200 is less than 10,000, the first part of the rule applies:

$$T(x) = .03x$$

$$T(9200) = .03(9200) = \$276$$

(b) \$30,000

Solution: Now we must find $T(30,000)$. Since 30,000 is greater than \$10,000, the second part of the rule applies:

$$\begin{aligned} T(x) &= 300 + .05(x - 10,000) \\ T(30,000) &= 300 + .05(30,000 - 10,000) \\ &= 300 + .05(20,000) \\ &= 300 + 1000 \\ &= \$1300 \end{aligned}$$

Appendix

We show that

$$\boxed{\sqrt{a^2} = |a|}$$

for any real number a. Indeed, suppose $a = -1$, then

$$\sqrt{a^2} = \sqrt{(-1)^2} = \sqrt{1} = 1 = |-1| = |a|$$

So, if $a = -1$, then $\sqrt{a^2} = |a|$. In general,

$$\sqrt{a^2} = |a|$$

for any negative number a. Note that if $a \geq 0$, then

$$\sqrt{a^2} = a$$

and $a = |a|$, since a is nonnegative. So,

$$\sqrt{a^2} = |a|$$

for *any* real number a.

COMPARE: Note that

$$\boxed{\sqrt[3]{a^3} = a}$$

for any real number a. Indeed,

$$\sqrt[3]{a^3} = (a^3)^{\frac{1}{3}} = a^{3 \cdot \frac{1}{3}} = a^1 = a$$

REMARK: Note that

$$\sqrt{a^2} = (a^2)^{\frac{1}{2}} \neq a^{2 \cdot \frac{1}{2}} = a^1 = a$$

if $a < 0$. In general, if $a \geq 0$, then

$$(a^n)^m = a^{nm} \qquad (*)$$

But $(*)$ might *not* be true if $a < 0$ and m or n are not integers.