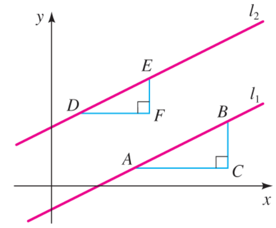
Parallel and Perpendicular Lines

Parallel Lines

Two nonvertical lines are parallel if and only if they have the same slope.



EXAMPLE: Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

Solution: We have

$$4x + 6y + 5 = 0 \quad \Longrightarrow \quad 6y = -4x - 5 \quad \Longrightarrow \quad y = \frac{-4x - 5}{6} = \frac{-4x}{6} - \frac{5}{6} = -\frac{2}{3}x - \frac{5}{6}$$

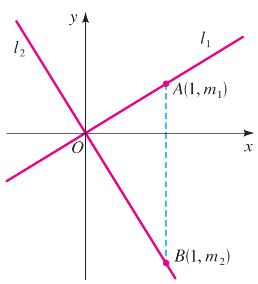
So the line has slope $m = -\dfrac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\dfrac{2}{3}$. From the point-slope form of the equation of a line, we get

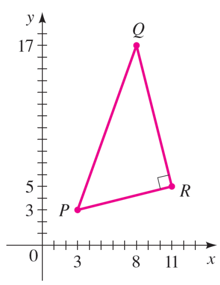
$$y - 2 = -\frac{2}{3}(x - 5)$$

Perpendicular Lines

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$





EXAMPLE: Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

Solution: The slopes of the lines containing PR and QR are, respectively,

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{11 - 3} = \frac{2}{8} = \frac{1}{4} \qquad \text{and} \qquad m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 17}{11 - 8} = \frac{-12}{3} = -4$$

Since $m_1 m_2 = -1$, these lines are perpendicular and so PQR is a right triangle. It is sketched in the Figure above (right).

EXAMPLE: Find an equation of the line that is perpendicular to the line $3x - 2y + 7 = 0$ and passes through the origin.

Solution: We first find the slope of the line $3x - 2y + 7 = 0$:

$$3x - 2y + 7 = 0 \quad \Longrightarrow \quad -2y = -3x - 7 \quad \Longrightarrow \quad y = \frac{-3x - 7}{-2} = \frac{-3x}{-2} - \frac{7}{-2} = \frac{3}{2}x + \frac{7}{2}$$

So the line has slope $m = \frac{3}{2}$. Thus, the slope of a perpendicular line is the negative reciprocal, that is, $-\frac{2}{3}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$y - 0 = -\frac{2}{3}(x - 0) \quad \Longrightarrow \quad y = -\frac{2}{3}x$$

EXAMPLE: Let $P(-3, 1)$ and $Q(5, 6)$ be two points in the coordinate plane. Find the perpendicular bisector of the line that contains P and Q.

Solution: The slope of the line that contains P and Q is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 6}{-3 - 5} = \frac{-5}{-8} = \frac{5}{8}$$

Therefore the slope of the perpendicular bisector of the line that contains P and Q is $-\frac{8}{5}$. We also need the midpoint of the segment PQ:

$$\left(\frac{-3 + 5}{2}, \frac{1 + 6}{2}\right) = \left(\frac{2}{2}, \frac{7}{2}\right) = \left(1, \frac{7}{2}\right)$$

It follows that an equation of the perpendicular bisector of the line that contains P and Q is

$$y - \frac{7}{2} = -\frac{8}{5}(x - 1)$$