Section 2.2 Equations of Lines

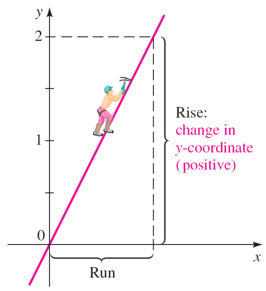
The Slope of a Line

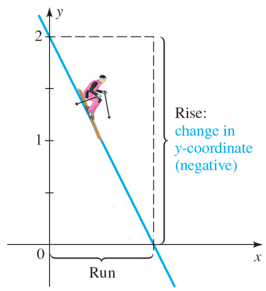
Slope of a Line

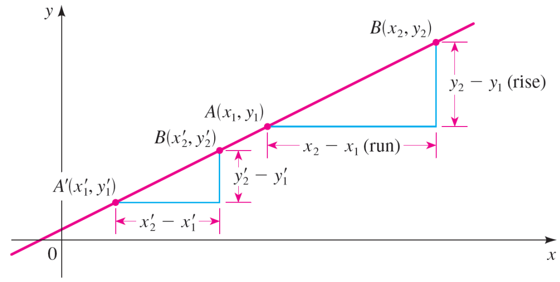
The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

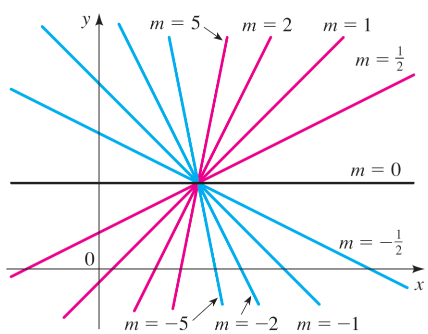
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.





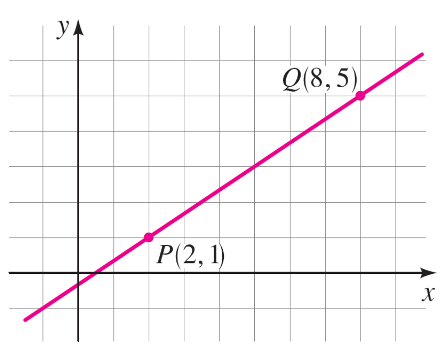




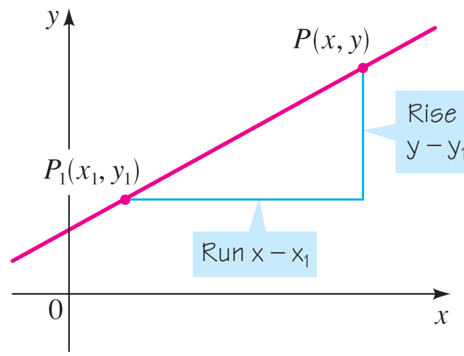
EXAMPLE: Find the slope of the line that passes through the points $P(2, 1)$ and $Q(8, 5)$.

Solution: We have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$



Point-Slope Form of the Equation of a Line



Point-Slope Form of the Equation of a Line

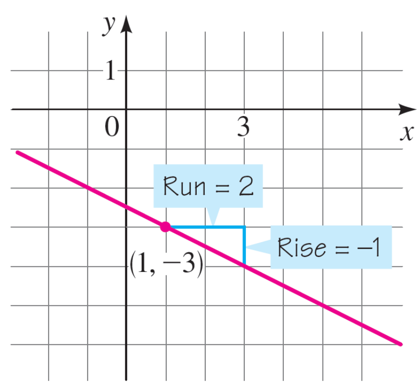
An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE: Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$ and sketch the line.

Solution: Using the point-slope form with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -3$, we obtain an equation of the line as

$$y + 3 = -\frac{1}{2}(x - 1)$$



EXAMPLE: Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

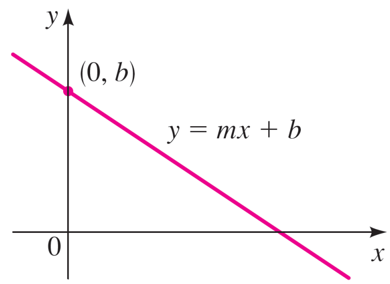
Solution: The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 2}{3 - (-1)} = \frac{-4 - 2}{3 + 1} = \frac{-6}{4} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\frac{3}{2}(x + 1)$$

Slope-Intercept Form of the Equation of a Line



Slope-Intercept Form of the Equation of a Line

An equation of the line that has slope m and y-intercept b is

$$y = mx + b$$

EXAMPLE: Find an equation of the line with slope 3 and y-intercept -2.

Solution: $y = mx + b = 3x + (-2) = 3x - 2$

EXAMPLE: Find an equation for the line that has x-intercept 6 and y-intercept 4.

Solution: Since the line passes through the points $(6, 0)$ and $(0, 4)$, the slope of this line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{0 - 6} = \frac{4}{-6} = -\frac{2}{3}$$

Since the y-intercept is 4, it follows that $b = 4$. Therefore the slope-intercept equation of the line is

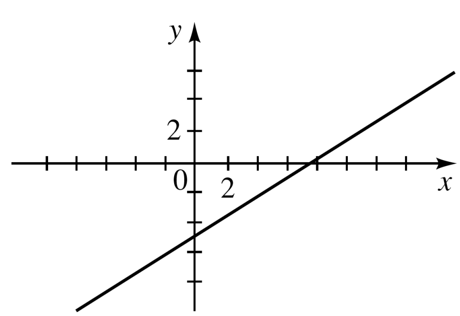
$$y = mx + b = -\frac{2}{3}x + 4$$

EXAMPLE: Write the linear equation $2x - 3y = 15$ in slope-intercept form, and sketch its graph. What are the slope and y-intercept?

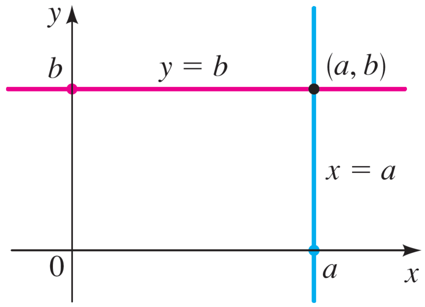
Solution: We have

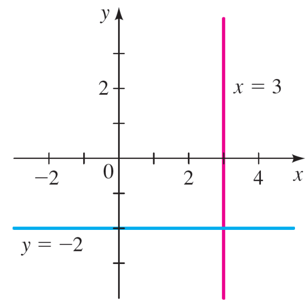
$$2x - 3y = 15 \quad \Longrightarrow \quad -3y = -2x + 15 \quad \Longrightarrow \quad y = \frac{-2x + 15}{-3} = \frac{-2x}{-3} + \frac{15}{-3} = \frac{2}{3}x - 5$$

Therefore the slope is $\frac{2}{3}$ and y-intercept is -5.



Vertical and Horizontal Lines





EXAMPLE:

(a) The graph of the equation $x = 3$ is a vertical line with x-intercept 3.

(b) The graph of the equation $y = -2$ is a horizontal line with y-intercept -2.

General Equation of a Line

General Equation of a Line

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

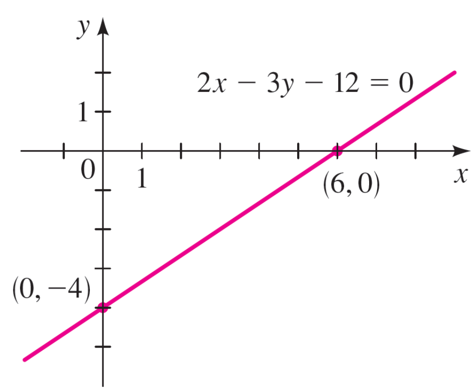
EXAMPLE: Sketch the graph of the equation $2x - 3y - 12 = 0$.

Solution: Since the equation is linear, its graph is a line. To draw the graph, it is enough to find any two points on the line. The intercepts are the easiest points to find.

x-intercept: Substitute $y = 0$, to get $2x - 12 = 0 \implies 2x = 12 \implies x = \dfrac{12}{2} = 6$

y-intercept: Substitute $x = 0$, to get $-3y - 12 = 0 \implies -3y = 12 \implies y = \dfrac{12}{-3} = -4$

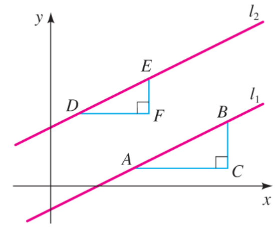
With these points we can sketch the graph in the Figure below.



Parallel and Perpendicular Lines

Parallel Lines

Two nonvertical lines are parallel if and only if they have the same slope.



EXAMPLE: Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

Solution: We have

$$4x + 6y + 5 = 0 \quad \Longrightarrow \quad 6y = -4x - 5 \quad \Longrightarrow \quad y = \frac{-4x-5}{6} = \frac{-4x}{6} - \frac{5}{6} = -\frac{2}{3}x - \frac{5}{6}$$

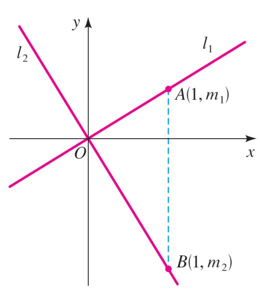
So the line has slope $m = -\dfrac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\dfrac{2}{3}$. From the point-slope form of the equation of a line, we get

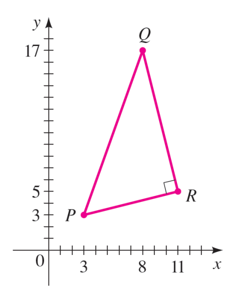
$$y - 2 = -\frac{2}{3}(x - 5)$$

Perpendicular Lines

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$





EXAMPLE: Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

Solution: The slopes of the lines containing PR and QR are, respectively,

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{11-3} = \frac{2}{8} = \frac{1}{4} \qquad \text{and} \qquad m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-17}{11-8} = \frac{-12}{3} = -4$$

Since $m_1 m_2 = -1$, these lines are perpendicular and so PQR is a right triangle. It is sketched in the Figure above (right).

EXAMPLE: Find an equation of the line that is perpendicular to the line $3x-2y+7=0$ and passes through the origin.

Solution: We first find the slope of the line $3x-2y+7=0$:

$$3x-2y+7=0 \quad \Longrightarrow \quad -2y=-3x-7 \quad \Longrightarrow \quad y=\frac{-3x-7}{-2}=\frac{-3x}{-2}-\frac{7}{-2}=\frac{3}{2}x+\frac{7}{2}$$

So the line has slope $m=\frac{3}{2}$. Thus, the slope of a perpendicular line is the negative reciprocal, that is, $-\frac{2}{3}$. Since the required line passes through $(0,0)$, the point-slope form gives

$$y-0=-\frac{2}{3}(x-0) \quad \Longrightarrow \quad y=-\frac{2}{3}x$$

EXAMPLE: Let $P(-3,1)$ and $Q(5,6)$ be two points in the coordinate plane. Find the perpendicular bisector of the line that contains P and Q.

Solution: The slope of the line that contains P and Q is

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-6}{-3-5}=\frac{-5}{-8}=\frac{5}{8}$$

Therefore the slope of the perpendicular bisector of the line that contains P and Q is $-\frac{8}{5}$. We also need the midpoint of the segment PQ:

$$\left(\frac{-3+5}{2},\frac{1+6}{2}\right)=\left(\frac{2}{2},\frac{7}{2}\right)=\left(1,\frac{7}{2}\right)$$

It follows that an equation of the perpendicular bisector of the line that contains P and Q is

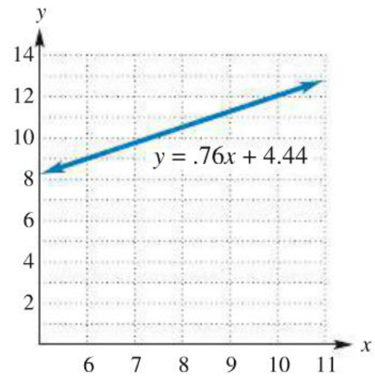
$$y-\frac{7}{2}=-\frac{8}{5}(x-1)$$

Applications

EXAMPLE: The world-wide sales (in billions of dollars) of men's razor blades can be approximated by the linear equation

$$y = .76x + 4.44$$

where $x = 6$ corresponds to the year 2006. The graph appears in the Figure below. (Data from: Wall Street Journal, April 12, 2012.)



(a) What were the approximate razor sales in 2011?

Solution: Substitute $x = 11$ in the equation and compute y:

$$y = .76x + 4.44$$

$$y = .76(11) + 4.44 = 12.80$$

The approximate sales in 2011 were \$12.8 billion.

(b) In what year did sales reach \$10.5 billion?

Solution: Substitute $y = 10.5$ in the equation and solve for x:

$$10.5 = .76x + 4.44$$

$$10.5 - 4.44 = .76x$$

$$x = \frac{10.5 - 4.44}{.76} = \frac{6.06}{.76} \approx 8.0$$

The year was 2008.