

Section 2.2 Equations of Lines

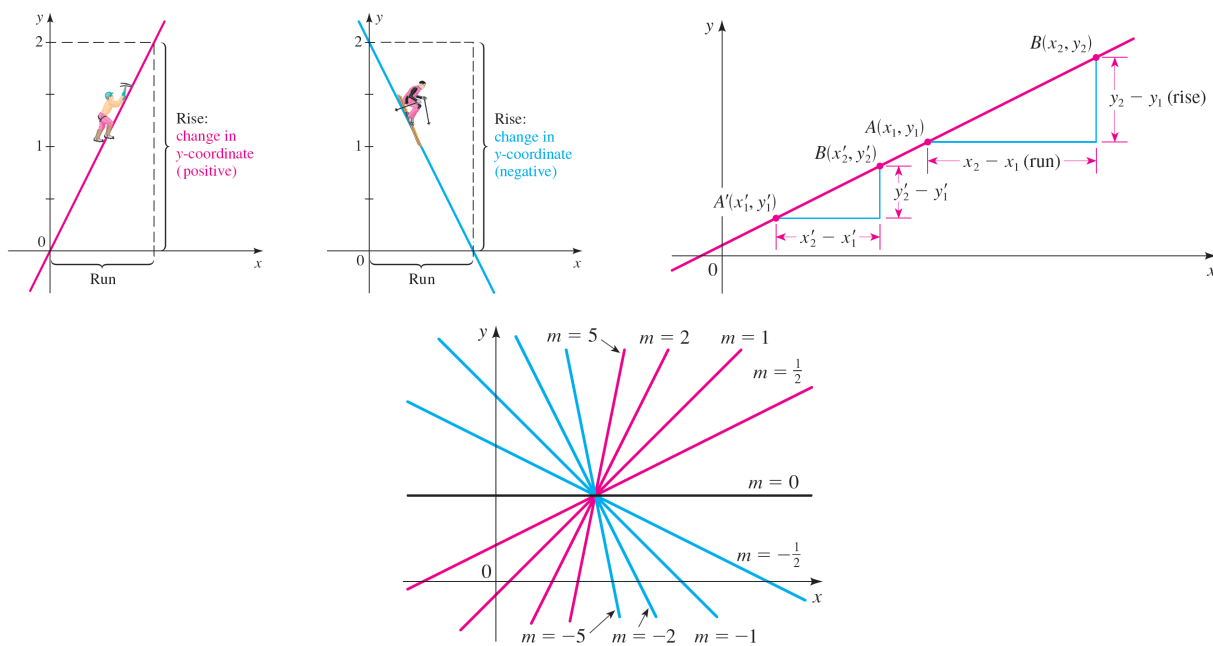
The Slope of a Line

Slope of a Line

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

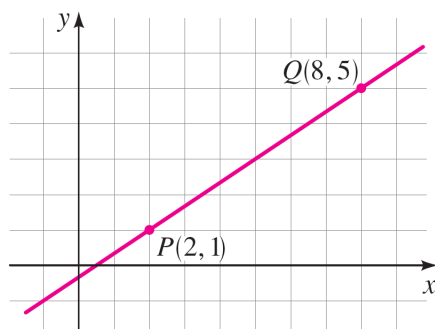
The slope of a vertical line is not defined.



EXAMPLE: Find the slope of the line that passes through the points $P(2, 1)$ and $Q(8, 5)$.

Solution: We have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$



EXAMPLE: Find the slope of the line that passes through the points $P(-2, -1)$ and $Q(-8, 5)$.

Solution: We have

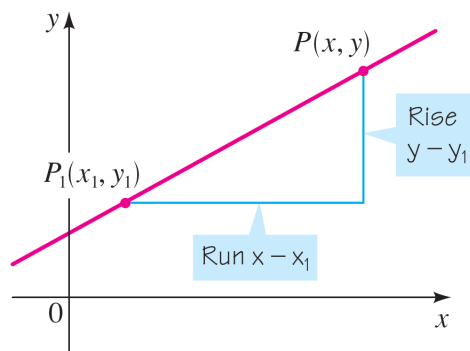
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-1)}{-8 - (-2)} = \frac{5 + 1}{-8 + 2} = \frac{6}{-6} = -1$$

EXAMPLE: Find the slope of the line that passes through the points $P(-3, 1)$ and $Q(5, 6)$.

Solution: We have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 1}{5 - (-3)} = \frac{6 - 1}{5 + 3} = \frac{5}{8}$$

Point-Slope Form of the Equation of a Line



From the picture above it follows that

$$\frac{y - y_1}{x - x_1} = m$$

which can be rewritten as

$$y - y_1 = m(x - x_1)$$

Point-Slope Form of the Equation of a Line

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE: Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$ and sketch the line.

EXAMPLE: Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$ and sketch the line.

Solution: Using the point-slope form with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -3$, we obtain an equation of the line as

$$y - (-3) = -\frac{1}{2}(x - 1)$$

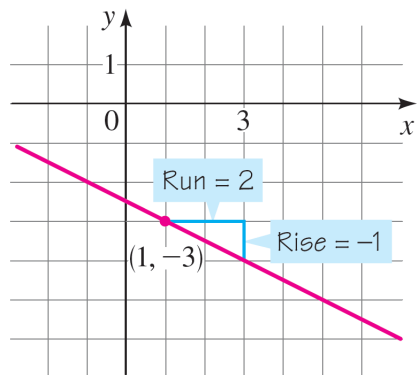
$$y + 3 = -\frac{1}{2}(x - 1)$$

$$2(y + 3) = -(x - 1)$$

$$2y + 6 = -x + 1$$

$$2y + 6 + x - 1 = 0$$

$$x + 2y + 5 = 0$$



EXAMPLE: Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

Solution: The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 2}{3 - (-1)} = \frac{-4 - 2}{3 + 1} = \frac{-6}{4} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\frac{3}{2}(x - (-1))$$

$$y - (-4) = -\frac{3}{2}(x - 3)$$

$$y - 2 = -\frac{3}{2}(x + 1)$$

$$y + 4 = -\frac{3}{2}(x - 3)$$

$$2(y - 2) = -3(x + 1)$$

$$2(y + 4) = -3(x - 3)$$

or

$$2y - 4 = -3x - 3$$

$$2y + 8 = -3x + 9$$

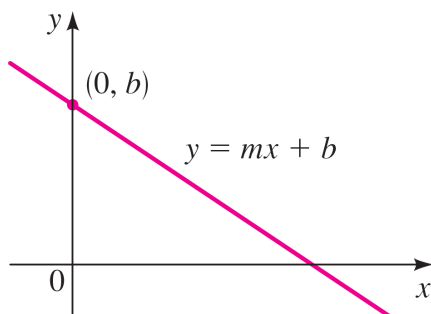
$$2y - 4 + 3x + 3 = 0$$

$$2y + 8 + 3x - 9 = 0$$

$$3x + 2y - 1 = 0$$

$$3x + 2y - 1 = 0$$

Slope-Intercept Form of the Equation of a Line



From the picture above it follows that

$$y - b = m(x - 0)$$

which can be rewritten as

$$y = mx + b$$

Slope-Intercept Form of the Equation of a Line

An equation of the line that has slope m and y -intercept b is

$$y = mx + b$$

EXAMPLE: Find an equation of the line with slope 3 and y -intercept -2 .

Solution: Since $m = 3$ and $b = -2$, from the slope-intercept form of the equation of a line we get

$$y = mx + b = 3x + (-2) = 3x - 2$$

EXAMPLE: Find an equation for the line that has x -intercept 6 and y -intercept 4.

Solution 1: Since the line passes through the points $(6, 0)$ and $(0, 4)$, the slope of this line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{0 - 6} = \frac{4}{-6} = -\frac{2}{3}$$

Since the y -intercept is 4, it follows that $b = 4$. Therefore the slope-intercept equation of the line is

$$y = mx + b = -\frac{2}{3}x + 4$$

Solution 2: Since the y -intercept is 4, it follows that the slope-intercept equation of the line is $y = mx + 4$. Since the x -intercept is 6, the line passes through the point $(6, 0)$, therefore

$$0 = m \cdot 6 + 4 \implies -4 = 6m \implies m = \frac{-4}{6} = -\frac{2}{3}$$

and the same result follows.

EXAMPLE: Find the slope and y -intercept of the line $3y - 2x = 1$.

EXAMPLE: Find the slope and y -intercept of the line $3y - 2x = 1$.

Solution: We first write the equation in the form $y = mx + b$:

$$3y - 2x = 1$$

$$3y = 2x + 1$$

$$y = \frac{2x + 1}{3} = \frac{2}{3}x + \frac{1}{3}$$

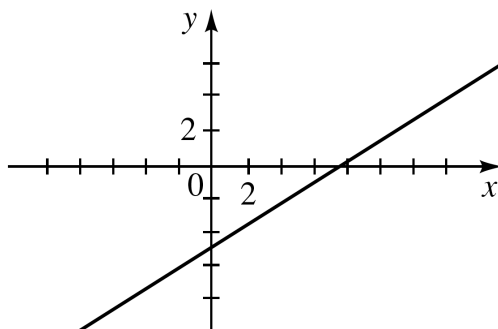
From the slope-intercept form of the equation of a line, we see that the slope is $m = \frac{2}{3}$ and the y -intercept is $b = \frac{1}{3}$.

EXAMPLE: Write the linear equation $2x - 3y = 15$ in slope-intercept form, and sketch its graph. What are the slope and y -intercept?

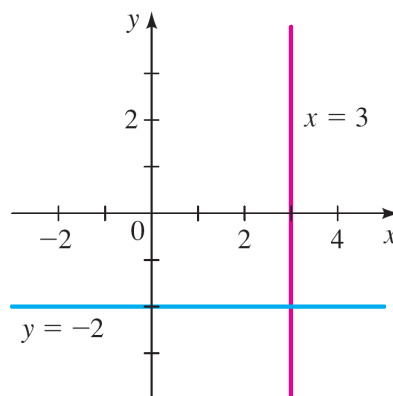
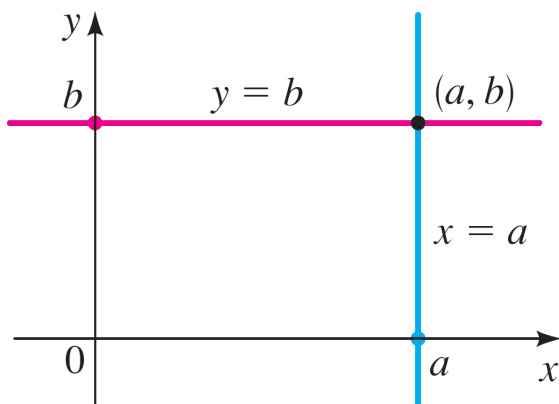
Solution: We have

$$2x - 3y = 15 \implies -3y = -2x + 15 \implies y = \frac{-2x + 15}{-3} = \frac{-2x}{-3} + \frac{15}{-3} = \frac{2}{3}x - 5$$

Therefore the slope is $\frac{2}{3}$ and y -intercept is -5 .



Vertical and Horizontal Lines



EXAMPLE:

- (a) The graph of the equation $x = 3$ is a vertical line with x -intercept 3.
- (b) The graph of the equation $y = -2$ is a horizontal line with y -intercept -2 .

General Equation of a Line

General Equation of a Line

The graph of every **linear equation**

$$Ax + By + C = 0 \quad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

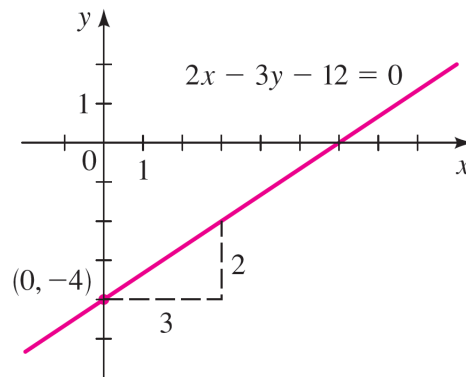
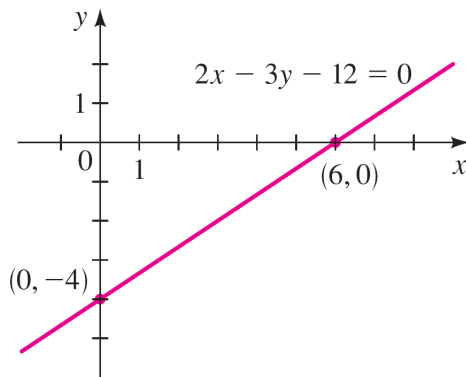
EXAMPLE: Sketch the graph of the equation $2x - 3y - 12 = 0$.

Solution 1: Since the equation is linear, its graph is a line. To draw the graph, it is enough to find any two points on the line. The intercepts are the easiest points to find.

$$x\text{-intercept: Substitute } y = 0, \text{ to get } 2x - 12 = 0 \implies 2x = 12 \implies x = \frac{12}{2} = 6$$

$$y\text{-intercept: Substitute } x = 0, \text{ to get } -3y - 12 = 0 \implies -3y = 12 \implies y = \frac{12}{-3} = -4$$

With these points we can sketch the graph in the Figure below (left).



Solution 2: We write the equation in slope-intercept form:

$$2x - 3y - 12 = 0$$

$$-3y = -2x + 12$$

$$y = \frac{-2x + 12}{-3} = \frac{-2x}{-3} + \frac{12}{-3} = \frac{2}{3}x - 4$$

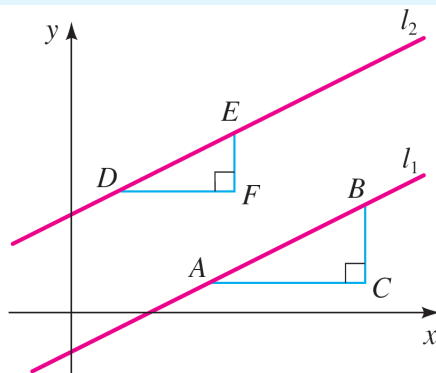
This equation is in the form $y = mx + b$, so the slope is $m = \frac{2}{3}$ and the y -intercept is $b = -4$. To sketch the graph, we plot the y -intercept, and then move 3 units to the right and 2 units up as shown in the Figure above (right).

REMARK: One can show that the vector with coordinates (A, B) is perpendicular to the line $Ax + By + C = 0$.

Parallel and Perpendicular Lines

Parallel Lines

Two nonvertical lines are parallel if and only if they have the same slope.



EXAMPLE: Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

Solution: We have

$$4x + 6y + 5 = 0$$

$$6y = -4x - 5$$

$$y = \frac{-4x - 5}{6} = \frac{-4x}{6} - \frac{5}{6} = -\frac{2}{3}x - \frac{5}{6}$$

So the line has slope $m = -\frac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\frac{2}{3}$. From the point-slope form of the equation of a line, we get

$$y - 2 = -\frac{2}{3}(x - 5)$$

$$3(y - 2) = -2(x - 5)$$

$$3y - 6 = -2x + 10$$

$$3y - 6 + 2x - 10 = 0$$

$$2x + 3y - 16 = 0$$

Thus, the equation of the required line is $2x + 3y - 16 = 0$.

EXAMPLE: Find an equation for the line that passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$.

EXAMPLE: Find an equation for the line that passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$.

Solution: We have

$$3x + y - 10 = 0 \implies y = -3x + 10$$

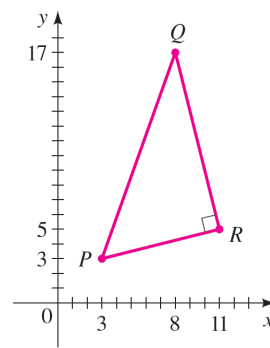
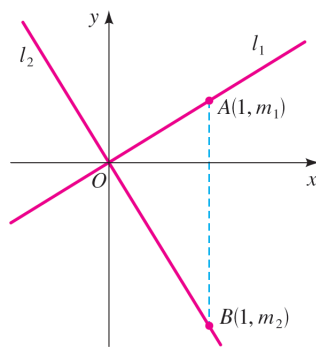
Therefore the slope of the line is -3 . It follows that an equation for the line that passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$ is

$$y - (-6) = -3(x - 3) \implies y + 6 = -3x + 9 \implies y = -3x + 3$$

Perpendicular Lines

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$



EXAMPLE: Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

Solution: The slopes of the lines containing PR and QR are, respectively,

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{11 - 3} = \frac{2}{8} = \frac{1}{4} \quad \text{and} \quad m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 17}{11 - 8} = \frac{-12}{3} = -4$$

Since $m_1 m_2 = -1$, these lines are perpendicular and so PQR is a right triangle. It is sketched in the Figure above (right).

EXAMPLE: Find an equation of the line that is perpendicular to the line $3x - 2y + 7 = 0$ and passes through the origin.

EXAMPLE: Find an equation of the line that is perpendicular to the line $3x - 2y + 7 = 0$ and passes through the origin.

Solution: We first find the slope of the line $3x - 2y + 7 = 0$:

$$3x - 2y + 7 = 0$$

$$-2y = -3x - 7$$

$$y = \frac{-3x - 7}{-2} = \frac{-3x}{-2} - \frac{7}{-2} = \frac{3}{2}x + \frac{7}{2}$$

So the line has slope $m = \frac{3}{2}$. Thus, the slope of a perpendicular line is the negative reciprocal, that is, $-\frac{2}{3}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$y - 0 = -\frac{2}{3}(x - 0)$$

$$y = -\frac{2}{3}x$$

EXAMPLE: Let $P(-3, 1)$ and $Q(5, 6)$ be two points in the coordinate plane. Find the perpendicular bisector of the line that contains P and Q .

Solution: The slope of the line that contains P and Q is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 6}{-3 - 5} = \frac{-5}{-8} = \frac{5}{8}$$

Therefore the slope of the perpendicular bisector of the line that contains P and Q is $-\frac{8}{5}$. We also need the midpoint of the segment PQ :

$$\left(\frac{-3 + 5}{2}, \frac{1 + 6}{2}\right) = \left(\frac{2}{2}, \frac{7}{2}\right) = \left(1, \frac{7}{2}\right)$$

It follows that an equation of the perpendicular bisector of the line that contains P and Q is

$$y - \frac{7}{2} = -\frac{8}{5}(x - 1)$$

$$y - \frac{7}{2} = -\frac{8}{5}x + \frac{8}{5}$$

so

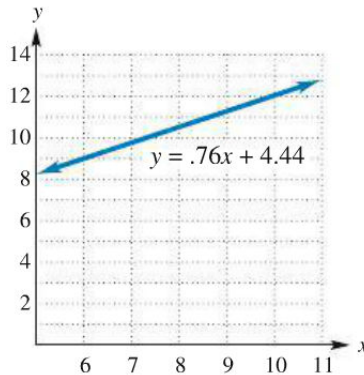
$$y = -\frac{8}{5}x + \frac{8}{5} + \frac{7}{2} = -\frac{8}{5}x + \frac{2 \cdot 8}{2 \cdot 5} + \frac{7 \cdot 5}{2 \cdot 5} = -\frac{8}{5}x + \frac{2 \cdot 8 + 7 \cdot 5}{2 \cdot 5} = -\frac{8}{5}x + \frac{16 + 35}{10} = -\frac{8}{5}x + \frac{51}{10}$$

Applications

EXAMPLE: The world-wide sales (in billions of dollars) of men's razor blades can be approximated by the linear equation

$$y = .76x + 4.44$$

where $x = 6$ corresponds to the year 2006. The graph appears in the Figure below. (Data from: Wall Street Journal, April 12, 2012.)



(a) What were the approximate razor sales in 2011?

Solution: Substitute $x = 11$ in the equation and compute y :

$$y = .76x + 4.44$$

$$y = .76(11) + 4.44 = 12.80$$

The approximate sales in 2011 were \$12.8 billion.

(b) In what year did sales reach \$10.5 billion?

Solution: Substitute $y = 10.5$ in the equation and solve for x :

$$10.5 = .76x + 4.44$$

$$10.5 - 4.44 = .76x$$

$$x = \frac{10.5 - 4.44}{.76} = \frac{6.06}{.76} \approx 8.0$$

The year was 2008.