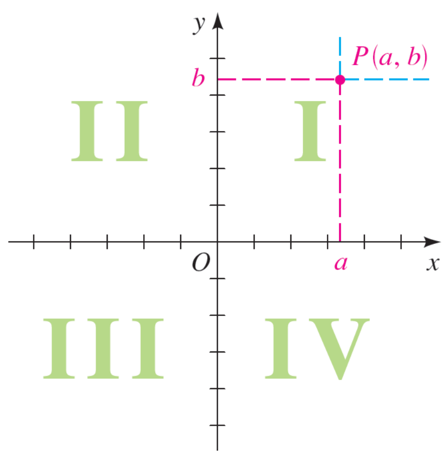
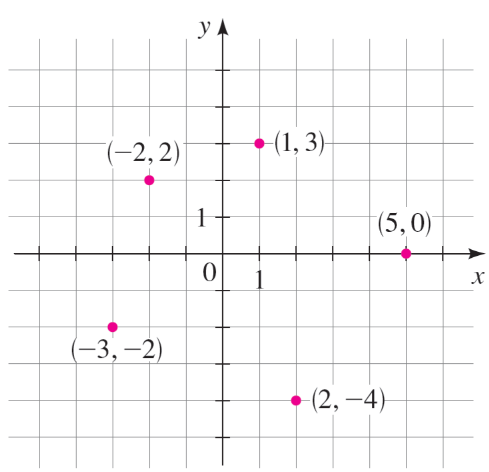
Section 2.1 Graphs

The Coordinate Plane

Just as points on a line can be identified with real numbers to form the coordinate line, points in a plane can be identified with ordered pairs of numbers to form the **coordinate plane** or **Cartesian plane**. To do this, we draw two perpendicular real lines that intersect at 0 on each line. Usually, one line is horizontal with positive direction to the right and is called the ***x*-axis**; the other line is vertical with positive direction upward and is called the ***y*-axis**. The point of intersection of the x-axis and the y-axis is the **origin *O*,** and the two axes divide the plane into four **quadrants**, labeled I, II, III, and IV in the Figure below. (The points *on* the coordinate axes are not assigned to any quadrant.)



Any point P in the coordinate plane can be located by a unique **ordered pair** of numbers (a, b), as shown in the Figure above. The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We can think of the coordinates of P as its "address," because they specify its location in the plane. Several points are labeled with their coordinates in the Figure below.



Graphs of Equations in Two Variables

The Graph of an Equation

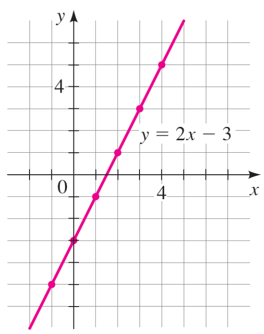
The **graph** of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

EXAMPLE: Sketch the graph of the equation $2x - y = 3$.

Solution: We first solve the given equation for y:

$$2x - y = 3 \quad \Longrightarrow \quad 2x = y + 3 \quad \Longrightarrow \quad 2x - 3 = y$$

This helps us calculate the y-coordinates in the following table and eventually sketch the graph.



x	$y = 2x - 3$	(x, y)
-1	-5	$(-1, -5)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	1	$(2, 1)$
3	3	$(3, 3)$
4	5	$(4, 5)$

General Equation of a Line

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

EXAMPLE: Sketch the graph of the equation $4x = 2 - 3y$.

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Solution: Since the equation $4x = 2 - 3y$ can be rewritten as $4x + 3y - 2 = 0$, it is linear. Therefore its graph is a line. To draw the graph, it is enough to find any two points on the line. In order to do that we first solve $4x = 2 - 3y$ for y:

$$4x = 2 - 3y \quad \Longrightarrow \quad 3y + 4x = 2 \quad \Longrightarrow \quad 3y = 2 - 4x \quad \Longrightarrow \quad y = \frac{2 - 4x}{3}$$

Now we can plug any two numbers in for x. For example,

Version 1: If $x = 0$, then

$$y = \frac{2 - 4(0)}{3} = \frac{2 - 0}{3} = \frac{2}{3}$$

Similarly, if $x = 1$, then

$$y = \frac{2 - 4(1)}{3} = \frac{2 - 4}{3} = -\frac{2}{3}$$

With the points $(0, 2/3)$ and $(1, -2/3)$ we can sketch the graph of the equation $4x = 2 - 3y$.

Version 2: If $x = -1$, then

$$y = \frac{2 - 4(-1)}{3} = \frac{2 + 4}{3} = \frac{6}{3} = 2$$

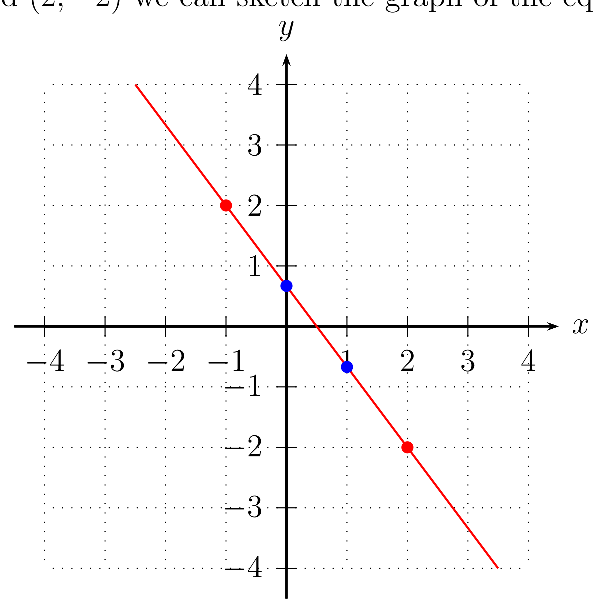
Similarly, if $x = 2$, then

$$y = \frac{2 - 4(2)}{3} = \frac{2 - 8}{3} = \frac{-6}{3} = -2$$

With the points $(-1, 2)$ and $(2, -2)$ we can sketch the graph of the equation $4x = 2 - 3y$.

Version 1

x	y
0	2/3
1	-2/3



Version 2

x	y
-1	2
2	-2

EXAMPLE: Sketch the graph of the equation $y = |x|$.

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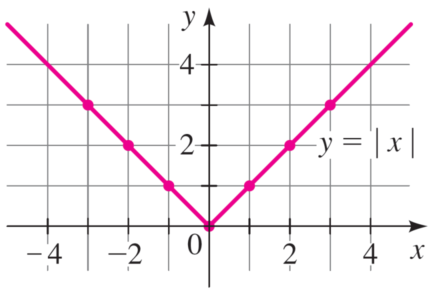
Solution: We make a table of values that helps us to sketch the graph of the equation.



x	$y = \|x\|$	(x, y)
-3	3	$(-3, 3)$
-2	2	$(-2, 2)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$
3	3	$(3, 3)$

EXAMPLE:

Sketch the graphs of the following functions.

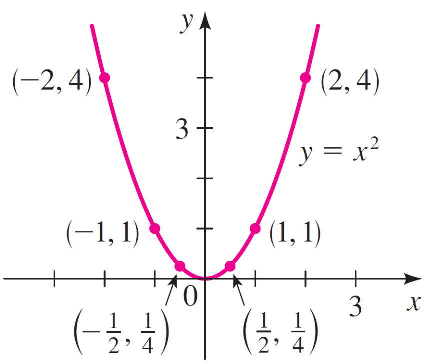
(a) $f(x) = x^2$ (b) $g(x) = x^3$ (c) $h(x) = \sqrt{x}$

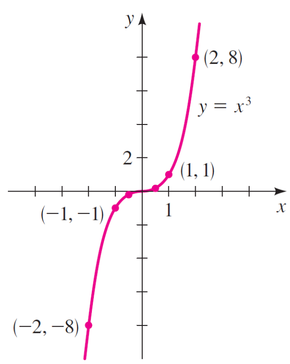
Solution: We first make a table of values. Then we plot the points given by the table and join them by a smooth curve to obtain the graph. The graphs are sketched in the Figures below.

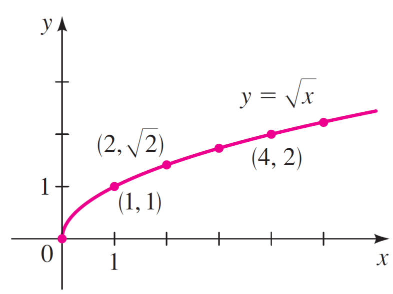
x	$f(x) = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

x	$g(x) = x^3$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
$-\frac{1}{2}$	$-\frac{1}{8}$
-1	-1
-2	-8

x	$h(x) = \sqrt{x}$
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{3}$
4	2
5	$\sqrt{5}$

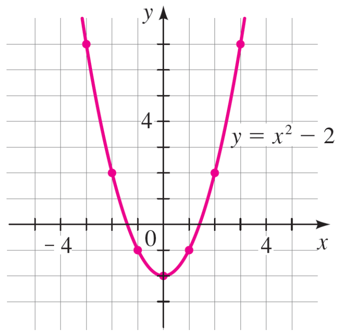






EXAMPLE: Sketch the graph of the equation $y = x^2 - 2$.

Solution: Plugging the numbers $0, \pm 1, \pm 2, \pm 3$ in for x, we find some of the points that satisfy the equation in the following table. In the Figure below we plot these points and then connect them by a smooth curve. A curve with this shape is called a *parabola*.

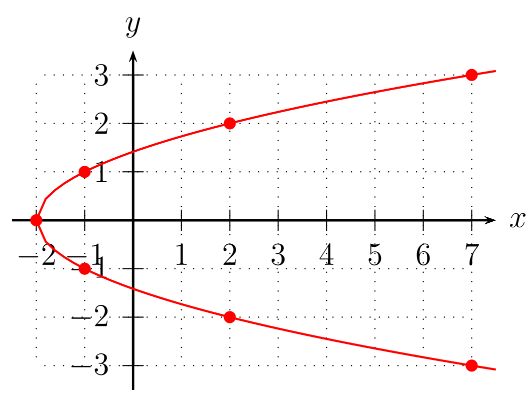


x	$y = x^2 - 2$	(x, y)
-3	7	$(-3, 7)$
-2	2	$(-2, 2)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
2	2	$(2, 2)$
3	7	$(3, 7)$

EXAMPLE: Sketch the graph of the equation $x = y^2 - 2$.

Solution: Plugging the numbers $0, \pm 1, \pm 2, \pm 3$ in for y, we find some of the points that satisfy the equation in the following table. In the Figure below we plot these points and then connect them by a smooth curve.

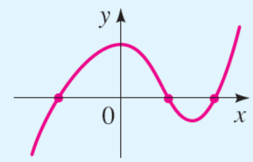
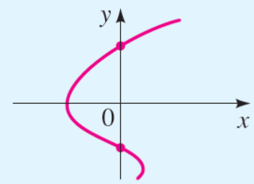
y	$x = y^2 - 2$	(x, y)
0	-2	(-2,0)
1	-1	(-1,1)
-1	-1	(-1,-1)
2	2	(2,2)
-2	2	(2,-2)
3	7	(7,3)
-3	7	(7,-3)



Intercepts

The x-coordinates of the points where a graph intersects the x-axis are called the **x-intercepts** of the graph and are obtained by setting $y = 0$ in the equation of the graph. The y-coordinates of the points where a graph intersects the y-axis are called the **y-intercepts** of the graph and are obtained by setting $x = 0$ in the equation of the graph.

Definition of Intercepts

Intercepts	How to find them	Where they are on the graph
***x*-intercepts:** The x-coordinates of points where the graph of an equation intersects the x-axis	Set $y = 0$ and solve for x	
***y*-intercepts:** The y-coordinates of points where the graph of an equation intersects the y-axis	Set $x = 0$ and solve for y	

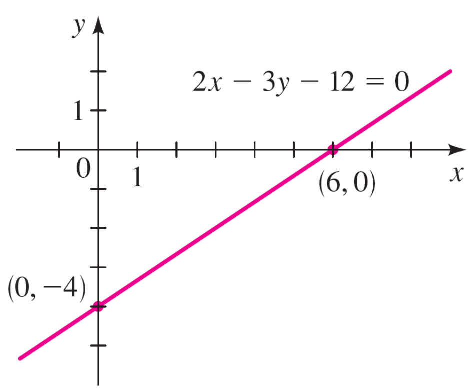
EXAMPLE: Sketch the graph of the equation $2x - 3y - 12 = 0$.

Solution: Since the equation is linear, its graph is a line. To draw the graph, it is enough to find any two points on the line. The intercepts are the easiest points to find.

$$x\text{-intercept: Substitute } y = 0, \text{ to get } 2x - 12 = 0 \implies 2x = 12 \implies x = \frac{12}{2} = 6$$

$$y\text{-intercept: Substitute } x = 0, \text{ to get } -3y - 12 = 0 \implies -3y = 12 \implies y = \frac{12}{-3} = -4$$

With these points we can sketch the graph in the Figure below.



EXAMPLE: Find the x- and y-intercepts of the graph of the equation $y = x^2 - 2$, and sketch the graph.

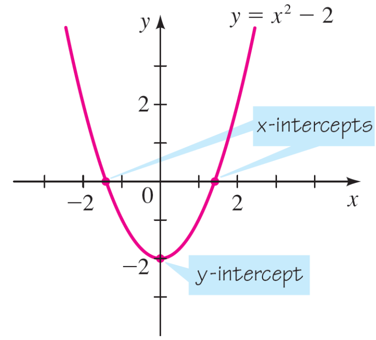
Solution: To find the x-intercepts, we set $y = 0$ and solve for x. Thus

$$0 = x^2 - 2 \quad \Longrightarrow \quad x^2 = 2 \quad \Longrightarrow \quad x = \pm\sqrt{2}$$

The x-intercepts are $\sqrt{2}$ and $-\sqrt{2}$.

To find the y-intercepts, we set $x = 0$ and find y. Thus $y = 0^2 - 2 = -2$. The y-intercept is -2. Now make a table, using both positive and negative values for x, and plot the corresponding points. These points suggest that the entire graph looks like the Figure below.

x	$y = x^2 - 2$
± 3	7
± 2	2
± 1	-1
0	-2



EXAMPLE: Find the x- and y-intercepts of the graph of the equation $y = x^2 - 2x - 8$, and sketch the graph.

Solution: To find the x-intercepts, we set $y = 0$ and solve for x. Thus

$$x^2 - 2x - 8 = 0$$

$$(x + 2)(x - 4) = 0$$

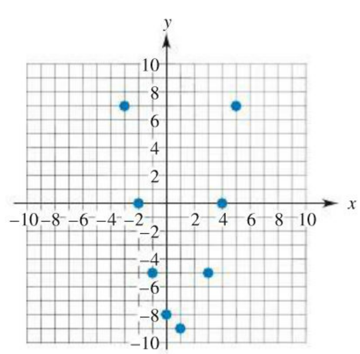
$$x + 2 = 0 \quad \text{or} \quad x - 4 = 0$$

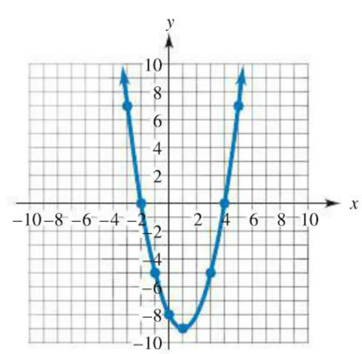
$$x = -2 \quad \text{or} \quad x = 4$$

The x-intercepts are 2 and 4.

To find the y-intercepts, we set $x = 0$ and find y. Thus $y = 0^2 - 2 \cdot 0 - 8 = -8$. The y-intercept is -8. Now make a table, using both positive and negative values for x, and plot the corresponding points. These points suggest that the entire graph looks like the Figure below.

x	$x^2 - 2x - 8$
−3	7
−2	0
−1	−5
0	−8
1	−9
3	−5
4	0
5	7

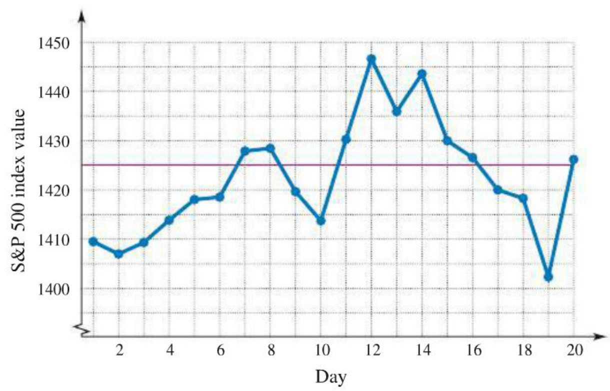




Graph Reading

Information is often given in graphical form, so you must be able to read and interpret graphs — that is, translate graphical information into statements in English.

EXAMPLE: Newspapers and websites summarize activity of the S&P 500 Index in graphical form. The results for the 20 trading days for the month of December, 2012 are displayed in the Figure below. The first coordinate of each point on the graph is the trading day in December, and the second coordinate represents the closing price of the S&P 500 on that day. (Data from: www.momingstar.com.)



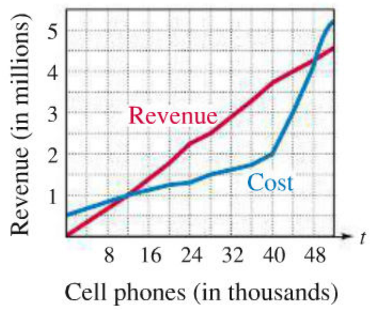
(a) What was the value of the S&P 500 Index on day 11 and day 17?

Solution: The point $(11, 1430)$ is on the graph, which means that the value of the index was 1430. The point $(17, 1420)$ is on the graph, which means that the value of the index was 1420 on that day.

(b) On what days was the value of the index above 1425?

Solution: Look for points whose second coordinates are greater than 1425 — that is, points that lie above the horizontal line through 1425 (shown in red in the Figure above). The first coordinates of these points are the days when the index value was above 1425. We see these days occurred on days 7, 8, 11, 12, 13, 14, 15, 16, and 20.

EXAMPLE: Monthly revenue and costs for the Webster Cell Phone Company are determined by the number t of phones produced and sold, as shown in the Figure below.



(a) How many phones should be produced each month if the company is to make a profit (assuming that all phones produced are sold)?

Solution: Recall that

$$\boxed{\text{Profit} = \text{Revenue} - \text{Cost}}$$

So the company makes a profit whenever revenue is greater than cost — that is, when the revenue graph is above the cost graph. The Figure above shows that this occurs between $t = 12$ and $t = 48$ — that is, when 12,000 to 48,000 phones are produced. If the company makes fewer than 12,000 phones, it will lose money (because costs will be greater than revenue.) It also loses money by making more than 48,000 phones. (One reason might be that high production levels require large amounts of overtime pay, which drives costs up too much.)

(b) Is it more profitable to make 40,000 or 44,000 phones?

Solution: On the revenue graph, the point with first coordinate 40 has second coordinate of approximately 3.7, meaning that the revenue from 40,000 phones is about 3.7 million dollars. The point with first coordinate 40 on the cost graph is $(40, 2)$, meaning that the cost of producing 40,000 phones is 2 million dollars. Therefore, the profit on 40,000 phones is about $3.7 - 2 = 1.7$ million dollars. For 44,000 phones, we have the approximate points $(44, 4)$ on the revenue graph and $(44, 3)$ on the cost graph. So the profit on 44,000 phones is $4 - 3 = 1$ million dollars. Consequently, it is more profitable to make 40,000 phones.