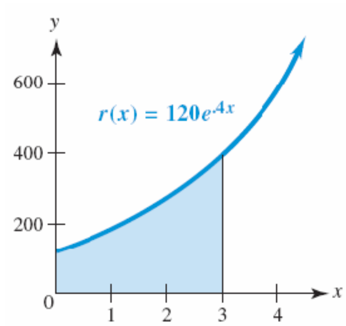
Section 13.5 Applications of Integrals

EXAMPLE: When you buy a used car, you may be offered the opportunity to buy a warranty that covers repair costs for a certain period. A company that provides such warranties uses current data and past experience to determine that the annual rate of repair costs for a particular model is given in year x by

$$r(x) = 120e^{.4x}$$

What will the total repair costs be for one year and for three years?



Solution: The total repair costs are represented by the area under the rate curve over the appropriate interval. For the first three years, the area is shown in the Figure above and is given by the integral

$$\int_0^3 120e^{.4x}dx = 120\int_0^3 e^{.4x}dx = 120\frac{1}{.4}e^{.4x}\Big|_0^3 = 300e^{.4x}\Big|_0^3 = 300e^{.4(3)} - 300e^{.4(0)}$$

$$= 300e^{1.2} - 300 \approx \$696.04$$

Similarly, the total repair costs for one year are

$$\int_0^1 120e^{.4x}dx = 300e^{.4x}\Big|_0^1 = 300e^{.4} - 300 \approx \$147.55$$

In order to make a profit, the company must charge more than \$147.55 for a one-year warranty and more than \$696.04 for a three-year warranty.

EXAMPLE: The rate at which revenue is generated (in billions of dollars per year) by McDonald's Corp. can be approximated by

$$f(x) = 1.1x + 14$$

where $x = 0$ corresponds to the year 2000. (Data from: www.morningstar.com.)

(a) What was the total revenue generated over the 5-year period from the beginning of 2008 to the end of 2012?

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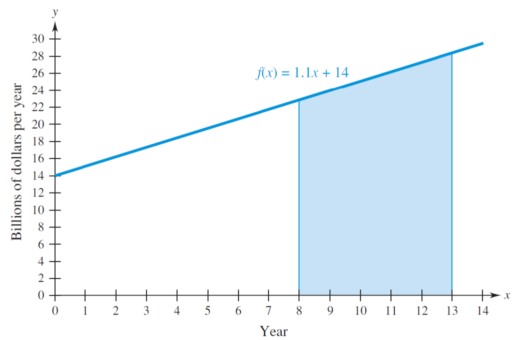
(a) What was the total revenue generated over the 5-year period from the beginning of 2008 to the end of 2012?

Solution: Since the end of 2012 coincides with the start of 2013, the total revenue generated is the area under the graph of $f(x)$ and above the x-axis from $x = 8$ to $x = 13$, as shown in the Figure below. This area is given by

$$\int_8^{13} (1.1x + 14)dx = \left(\frac{1.1}{2}x^2 + 14x\right)\Bigg|_8^{13} = \left(.55x^2 + 14x\right)\Bigg|_8^{13}$$

$$= \left(.55(13)^2 + 14(13)\right) - \left(.55(8)^2 + 14(8)\right) = 127.75$$

So the total revenue generated in this period was approximately \$127.75 billion.



(b) In what year (starting in 2000) did the total revenue reach \$195 billion?

Solution: The total revenue from $x = 0$ to $x = t$ is given by $\int_0^t (1.1x + 14)dx$. So we must find t such that

$$\int_0^t (1.1x + 14)dx = 195$$

$$\left(\frac{1.1}{2}x^2 + 14x\right)\Bigg|_0^t = 195$$

$$\left(.55x^2 + 14x\right)\Bigg|_0^t = 195$$

$$[.55(t)^2 + 14(t)] - [.55(0)^2 + 14(0)] = 195$$

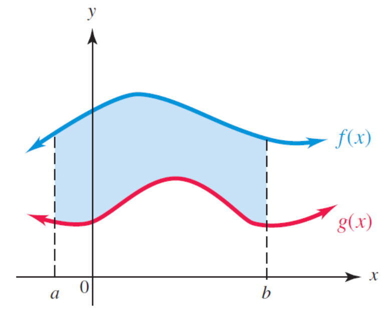
$$.55t^2 + 14t = 195$$

$$.55t^2 + 14t - 195 = 0$$

Using the quadratic formula, we obtain the solutions $t = 10$ and $t \approx -35.45$. Only the positive solution is meaningful here. Hence, total revenue reached \$195 billion when $t = 10$ (at the beginning of 2010).

Area between Two Curves

In some applications, it is necessary to find the area between two curves. For example, the area between the graphs of $f(x)$ and $g(x)$ from $x = a$ to $x = b$ in the Figure on the right is shaded. This area is the area under the graph of $f(x)$ *minus* the area under the graph of $g(x)$, that is,



$$\int_a^b f(x)dx - \int_a^b g(x)dx$$

which can be written as

$$\int_a^b [f(x) - g(x)]dx$$

Similar arguments in other cases (even when the graphs are not above the x-axis) produce the following important result.

Area between Two Curves

If f and g are continuous functions and $f(x) \geq g(x)$ on the interval $[a, b]$, then the area between the graphs of $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE: The rate at which revenue is generated from sales (in millions of dollars per year) for General Mills, Inc., can be approximated by

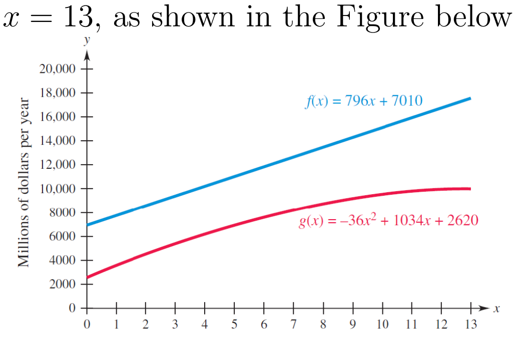
$$f(x) = 796x + 7010$$

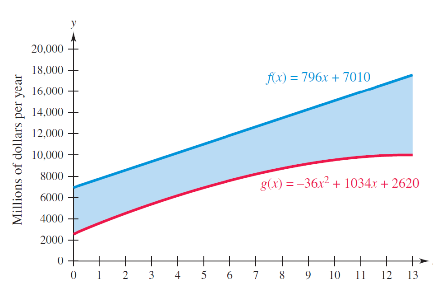
where $x = 0$ corresponds to the year 2000. The rate at which costs are incurred (in millions of dollars per year) can be approximated by

$$g(x) = -36x^2 + 1034x + 2620$$

Find the total profit earned between $x = 0$ and $x = 13$. (Data from: www.morningstar.com.)

Solution: The rate of revenue and rate of cost functions are graphed in the Figure below (left). Total revenue during this period is the area under the rate of revenue curve, and total cost is the area under the rate of cost curve. So the profit is the area between the two curves from $x = 0$ to $x = 13$, as shown in the Figure below (right).





This area is

$$\int_0^{13} [f(x) - g(x)]dx = \int_0^{13} [(796x + 7010) - (-36x^2 + 1034x + 2620)]dx$$

$$= \int_0^{13} (796x + 7010 + 36x^2 - 1034x - 2620)dx$$

$$= \int_0^{13} (36x^2 - 238x + 4390)dx$$

$$= \left(36\frac{x^3}{3} - 238\frac{x^2}{2} + 4390x\right)\Bigg|_0^{13}$$

$$= \left(12x^3 - 119x^2 + 4390x\right)\Bigg|_0^{13}$$

$$= \left(12(13)^3 - 119(13)^2 + 4390(13)\right) - 0$$

$$= 63,323$$

The profit was approximately \$63,323,000,000 or \$63.323 billion.

If revenue, cost, and profit are denoted by $R(x)$, $C(x)$, and $P(x)$, respectively, then

$$P(x) = R(x) - C(x)$$

so that

$$P'(x) = R'(x) - C'(x)$$

As we saw in Chapter 12, maximal profit occurs at a value of x for which $P'(x) = 0$. The previous equation shows that $P'(x) = 0$ exactly when $R'(x) = C'(x)$. In other words, maximal profit occurs when marginal revenue and marginal cost are equal.

EXAMPLE: A manufacturer produces a fabric similar to nylon. The product has been selling well, with a marginal revenue (in hundreds of dollars per hundred bolts) given by

$$R'(x) = -.3x^2 + 9x + 11$$

where x is measured in hundreds of bolts of fabric. The marginal cost (in hundreds of dollars per hundred bolts) is given by

$$C'(x) = 2x + 6$$

(a) To maximize profit, how many bolts should the company produce?

Solution: The manufacturer should continue production until the marginal costs equal the marginal revenue. Find this point by solving the equation $R'(x) = C'(x)$ as follows:

$$-.3x^2 + 9x + 11 = 2x + 6$$

$$-3x^2 + 90x + 110 = 20x + 60$$

$$-3x^2 + 90x + 110 - 20x - 60 = 0$$

$$-3x^2 + 70x + 50 = 0$$

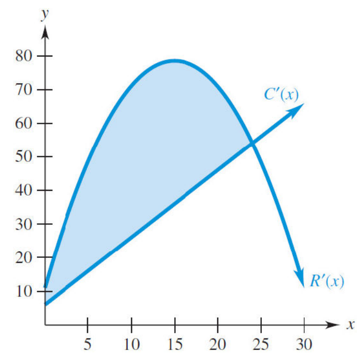
Find the positive solution by using the quadratic formula with $a = -3, b = 70$, and $c = 50$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-70 \pm \sqrt{(70)^2 - 4(-3)(50)}}{2(-3)} \approx 24.0$$

To maximize profit, the company should produce 2400 bolts of the fabric.

(b) What will the profit be on 2400 bolts?

Solution: To find the profit on 2400 bolts, find the area between the graphs of the marginal revenue and marginal cost functions, shown in the Figure below.



The area (profit) is

$$\begin{aligned}
\text{Profit} &= \int_0^{24} [R'(x) - C'(x)]dx \\
&= \int_0^{24} [(-.3x^2 + 9x + 11) - (2x + 6)]dx \\
&= \int_0^{24} (-.3x^2 + 9x + 11 - 2x - 6)dx \\
&= \int_0^{24} (-.3x^2 + 7x + 5)dx \\
&= \left(-.3\frac{x^3}{3} + 7\frac{x^2}{2} + 5x\right)\Bigg|_0^{24} \\
&= \left(-.1x^3 + \frac{7}{2}x^2 + 5x\right)\Bigg|_0^{24} \\
&= \left(-.1(24)^3 + \frac{7}{2}(24)^2 + 5(24)\right) - 0 \\
&= 753.6
\end{aligned}$$

The profit on 2400 bolts will be \$75,360.

EXAMPLE: A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time x according to the rate-of-savings function

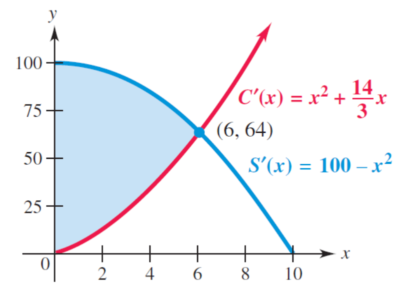
$$S'(x) = 100 - x^2$$

where x is measured in years and $S'(x)$ in thousands of dollars per year. At the same time, the cost of operating the new process increases with time x according to the rate-of-cost function (in thousands of dollars per year)

$$C'(x) = x^2 + \frac{14}{3}x$$

(a) To maximize its savings, for how many years should the company use the new process?

Solution: The Figure on the right shows the graphs of the rate-of-savings and the rate-of-cost functions. As was the case in the previous Example, maximum savings will occur when the rate of savings equals the rate of cost, that is, the time at which these graphs intersect.



The graphs intersect when

$$S'(x) = C'(x)$$

$$100 - x^2 = x^2 + \frac{14}{3}x$$

$$0 = x^2 + \frac{14}{3}x - 100 + x^2$$

$$0 = 2x^2 + \frac{14}{3}x - 100$$

$$0 = 6x^2 + 14x - 300$$

$$0 = 3x^2 + 7x - 150$$

To solve $3x^2 + 7x - 150 = 0$ we can either factor

$$3x^2 + 7x - 150 = 0$$

$$(x - 6)(3x + 25) = 0$$

$$x - 6 = 0 \quad \text{or} \quad 3x + 25 = 0$$

$$x = 6 \qquad\qquad 3x = -25$$

$$x = -\frac{25}{3}$$

or use the quadratic formula with $a = 3, b = 7$, and $c = -150$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-150)}}{2(3)} = \frac{-7 \pm \sqrt{1849}}{6} = \frac{-7 \pm 43}{6}$$

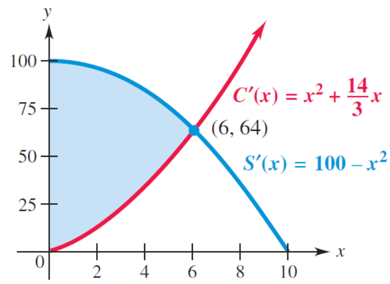
so

$$x = \frac{-7 + 43}{6} = \frac{36}{6} = 6 \qquad \text{or} \qquad x = \frac{-7 - 43}{6} = \frac{-50}{6} = -\frac{25}{3}$$

Only 6 is a meaningful solution here. The company should use the new process for 6 years.

(b) Taking the additional costs into account, how much will the company actually save during the 6-year period?

Solution: The actual savings (or net savings) are the difference between the savings and the additional costs. Total savings are given by the area under the rate-of-savings curve, and total additional costs are given by the area under the rate-of-cost curve, from $x = 0$ to $x = 6$. So total net savings are given by the difference between these two areas — that is, by the shaded area in the Figure below.



This area is given by the integral

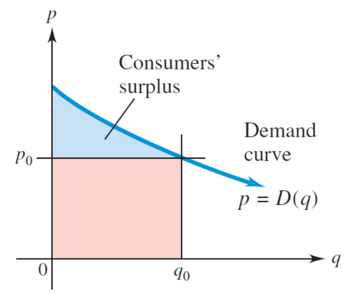
$$
\begin{aligned}
\text{Net savings } &= \int_0^6 [S'(x) - C'(x)]dx \\
&= \int_0^6 \left[(100 - x^2) - \left(x^2 + \frac{14}{3}x\right)\right] dx \\
&= \int_0^6 \left(100 - x^2 - x^2 - \frac{14}{3}x\right) dx \\
&= \int_0^6 \left(100 - \frac{14}{3}x - 2x^2\right) dx \\
&= \left(100x - \frac{14}{3}\frac{x^2}{2} - 2\frac{x^3}{3}\right)\Bigg|_0^6 \\
&= \left(100x - \frac{7}{3}x^2 - \frac{2}{3}x^3\right)\Bigg|_0^6 \\
&= \left(100(6) - \frac{7}{3}(6)^2 - \frac{2}{3}(6)^3\right) - 0 \\
&= 372
\end{aligned}
$$

The company will save a total of \$372,000 over the 6-year period.

Consumers' and Producers' Surplus

The market determines the price at which a product is sold. As indicated earlier, the point of intersection of the demand curve and the supply curve for a product gives the equilibrium price. At the equilibrium price, consumers will purchase the same amount of the product that the manufacturers want to sell. Some consumers, however, will be willing to spend more for an item than the equilibrium price. The total of the differences between the equilibrium price of the item and the higher prices all those individuals would be willing to pay is called the **consumers' surplus**.

In the Figure on the right, the (red and blue) colored area under the demand curve is the total amount consumers are willing to spend for q_0 items. The red area under the line $y = p_0$ shows the total amount consumers actually will spend at the equilibrium price of p_0. The blue area represents the consumers' surplus. As the figure suggests, the consumers' surplus is given by the area between the two curves $p = D(q)$ and $p = p_0$, so its value can be found with a definite integral as follows.

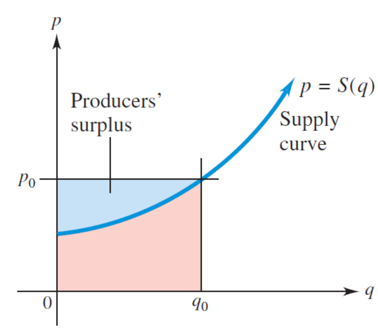


Consumers' Surplus

If $D(q)$ is a demand function with equilibrium price p_0 and equilibrium demand q_0, then

$$\textbf{Consumers' surplus} = \int_0^{q_0} [D(q) - p_0]\, dq.$$

Similarly, if some manufacturers would be willing to supply a product at a price *lower* than the equilibrium price p_0, the total of the differences between the equilibrium price and the lower prices at which the manufacturers would sell the product is called the **producers' surplus**. The Figure on the right shows the (red) area under the supply curve from $q = 0$ to $q = q_0$, which is the minimum total amount the manufacturers are willing to realize from the sale of q_0 items. The (red and blue) total area under the line $p = p_0$ is the amount actually realized. The difference between these two areas, the producers' surplus (blue), is also given by a definite integral.



Producers' Surplus

If $S(q)$ is a supply function with equilibrium price p_0 and equilibrium supply q_0, then

$$\textbf{Producers' surplus} = \int_0^{q_0} [p_0 - S(q)]\, dq.$$

EXAMPLE: Suppose the price (in dollars per ton) for oat bran is

$$D(q) = 900 - 20q - q^2$$

when the demand for the product is q tons. Also, suppose the function

$$S(q) = q^2 + 10q$$

gives the price (in dollars per ton) when the supply is q tons. Find the consumers' surplus and the producers' surplus.

Solution: We begin by finding the equilibrium quantity by setting the two equations equal:

$$\begin{aligned} D(q) &= S(q) \\ 900 - 20q - q^2 &= q^2 + 10q \\ 0 &= q^2 + 10q - 900 + 20q + q^2 \\ 0 &= 2q^2 + 30q - 900 \\ 0 &= q^2 + 15q - 450 \end{aligned}$$

To solve $q^2 + 15q - 450 = 0$ we can either factor

$$\begin{aligned} q^2 + 15q - 450 &= 0 \\ (q - 15)(q + 30) &= 0 \\ q - 15 = 0 \quad &\text{or} \quad q + 30 = 0 \\ q = 15 \qquad & \qquad q = -30 \end{aligned}$$

or use the quadratic formula with $a = 1, b = 15$, and $c = -450$:

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-15 \pm \sqrt{(15)^2 - 4(1)(-450)}}{2(1)} = \frac{-15 \pm \sqrt{2025}}{2} = \frac{-15 \pm 45}{2}$$

so

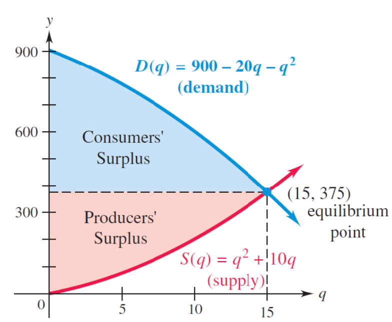
$$q = \frac{-15 + 45}{2} = \frac{30}{2} = 15 \quad \text{or} \quad q = \frac{-15 - 45}{2} = \frac{-60}{2} = -30$$

The only positive solution of this equation is $q = 15$. At the equilibrium point where the supply and demand are both $q_0 = 15$ tons, the price is

$$\text{Equilibrium price } = p_0 = S(q_0) = S(15) = 15^2 + 10(15) = 375$$

or \$375. Verify that the same answer is found by computing $D(15)$. The consumers' surplus, represented by the blue area shown in the Figure on the right, is

$$\text{Consumers' surplus } = \int_0^{q_0} [D(q) - p_0] dq$$



Evaluating this definite integral gives

$$\begin{aligned}
\text{Consumers' surplus } &= \int_0^{q_0} [D(q) - p_0]dq = \int_0^{15} [(900 - 20q - q^2) - 375]dq \\
&= \int_0^{15} (525 - 20q - q^2)dq \\
&= \left(525q - 20\frac{q^2}{2} - \frac{q^3}{3}\right)\Bigg|_0^{15} \\
&= \left(525q - 10q^2 - \frac{1}{3}q^3\right)\Bigg|_0^{15} \\
&= \left(525(15) - 10(15)^2 - \frac{1}{3}(15)^3\right) - 0 \\
&= 4500
\end{aligned}$$

Here, the consumers' surplus is \$4500. The producers' surplus (the red area in the Figure above) is given by

$$\begin{aligned}
\text{Producers' surplus } &= \int_0^{q_0} [p_0 - S(q)]dq = \int_0^{15} [375 - (q^2 + 10q)]dq \\
&= \int_0^{15} (375 - q^2 - 10q)dq \\
&= \left(375q - \frac{q^3}{3} - 10\frac{q^2}{2}\right)\Bigg|_0^{15} \\
&= \left(375q - \frac{1}{3}q^3 - 5q^2\right)\Bigg|_0^{15} \\
&= \left(375(15) - \frac{1}{3}(15)^3 - 5(15)^2\right) - 0 \\
&= 3375
\end{aligned}$$

The producers' surplus is \$3375.