Section 13.4 The Fundamental Theorem of Calculus

THEOREM (The Fundamental Theorem Of Calculus): If f is continuous on $[a, b]$, then

$$\boxed{\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b}$$

where F is any antiderivative of f, that is $F' = f$.

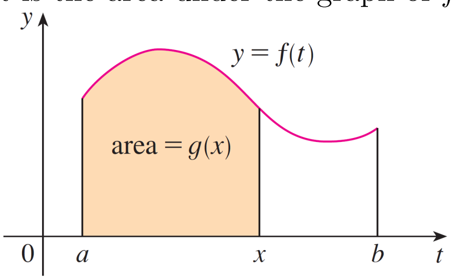
Proof (Sketch): Put

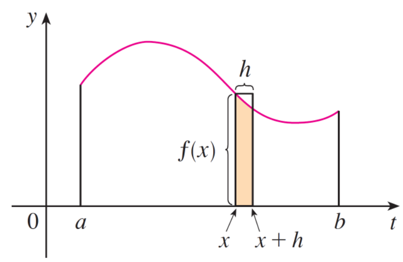
$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

Part I: We first show that g is an antiderivative of f, that is

$$g'(x) = f(x)$$

for $a < x < b$. Indeed, we first observe that, for $h > 0$, $g(x+h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of f from x to $x+h$ (the gold area).





For small h you can see that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h:

$$g(x+h) - g(x) \approx hf(x) \qquad \Longrightarrow \qquad \frac{g(x+h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

The fact that this is true, even when f is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

Part II: By Part I, $g(x)$ is an antiderivative of $f(x)$. Therefore any other antiderivative $F(x)$ of $f(x)$ can be written as

$$F(x) = g(x) + C = \int_a^x f(t)dt + C$$

It follows that

$$F(a) = \int_a^a f(t)dt + C = 0 + C = C \qquad \Longrightarrow \qquad F(b) = \int_a^b f(t)dt + C = \int_a^b f(t)dt + F(a)$$

thus

$$F(b) = \int_a^b f(t)dt + F(a) \qquad \Longrightarrow \qquad F(b) - F(a) = \int_a^b f(t)dt \ \blacksquare$$

REMARKS:

1. The fundamental theorem applies to every continuous function $f(x)$. It does not require that $f(x) \geq 0$.

2. If the antiderivative $F(x)$ is replaced by $F(x)+C$ for any constant C, the conclusion of the fundamental theorem is the same because C is eliminated in the final answer:

$$\int_a^b f(x)dx = (F(b)+C) - (F(a)+C) = F(b) + C - F(a) - C = F(b) - F(a)$$

3. The variable used in the integrand does not matter. Each of the following integrals represents the number $F(b) - F(a)$:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

4. The definition of $\int_a^b f(x)dx$ assumed that $a < b$ — that is, that the lower limit of integration is the smaller number. When $a > b$, we make this definition:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

For example, $\int_3^1 x^2dx = -\int_1^3 x^2dx$. Similarly, when the limits of integration are the same, we define $\int_a^a f(x)dx$ to be 0. The fundamental theorem is valid for such integrals.

Properties of the Definite Integral

1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

2. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

3. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

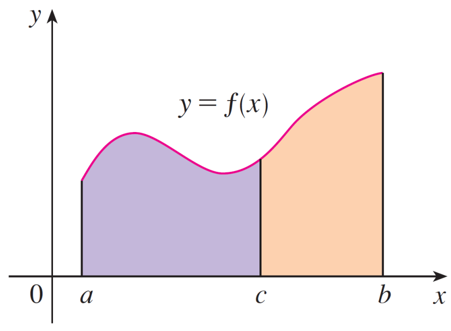


Table Of Indefinite Integrals

$\int cf(u)du = c\int f(u)du$	$\int [f(u) \pm g(u)]du = \int f(u)du \pm \int g(u)du$
$\int cdu = cu + C$	$\int udu = \frac{u^2}{2} + C$
$\int u^n du = \frac{u^{n+1}}{n+1} + C \ (n \neq -1)$	$\int \frac{1}{u}du = \ln\|u\| + C$
$\int e^u du = e^u + C$	$\int e^{ku}du = \frac{1}{k}e^{ku} + C$

EXAMPLES: Evaluate each of the given integrals.

(a) $\int\limits_2^3 7z^4 dz$

Solution: We have

$$\int\limits_2^3 7z^4 dz = 7\int\limits_2^3 z^4 dz = 7\left.\frac{z^{4+1}}{4+1}\right|_2^3 = 7\left.\frac{z^5}{5}\right|_2^3 = 7\left(\frac{3^5}{5} - \frac{2^5}{5}\right) = \frac{1477}{5}$$

(b) $\int\limits_0^1 8e^{4t} dt$

Solution: We have

$$\int\limits_0^1 8e^{4t} dt = 8\int\limits_0^1 e^{4t} dt = 8\left.\frac{1}{4}e^{4t}\right|_0^1 = \left.2e^{4t}\right|_0^1 = 2\left(e^{4(1)} - e^{4(0)}\right) = 2\left(e^4 - e^0\right) = 2\left(e^4 - 1\right)$$

(c) $\int\limits_4^9 \frac{1}{2}\sqrt{x} dx$

Solution: We have

$$\begin{aligned}\int\limits_4^9 \frac{1}{2}\sqrt{x} dx = \frac{1}{2}\int\limits_4^9 x^{1/2} dx = \frac{1}{2}\left.\frac{x^{1/2+1}}{1/2+1}\right|_4^9 &= \frac{1}{2}\left.\frac{x^{3/2}}{3/2}\right|_4^9 \\ &= \frac{1}{2}\left.\frac{2}{3}x^{3/2}\right|_4^9 = \left.\frac{1}{3}x^{3/2}\right|_4^9 = \frac{1}{3}\left(9^{3/2} - 4^{3/2}\right) \\ &= \frac{1}{3}\left(9^{(1/2)3} - 4^{(1/2)3}\right) \\ &= \frac{1}{3}\left((9^{1/2})^3 - (4^{1/2})^3\right) \\ &= \frac{1}{3}\left(3^3 - 2^3\right) \\ &= \frac{19}{3}\end{aligned}$$

(d) $\int\limits_{-3}^{-2} \frac{1}{x} dx$

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Solution: We have

$$\int_{-3}^{-2} \frac{1}{x} dx = \ln|x| \Big|_{-3}^{-2} = \ln|-2| - \ln|-3| = \ln 2 - \ln 3 = -0.4054...$$

(e) $\int_{1}^{4} (t^2 - t - 6) dt$

Solution: We have

$$\int_{1}^{4} (t^2 - t - 6) dt = \int_{1}^{4} t^2 dt - \int_{1}^{4} t dt - \int_{1}^{4} 6 dt = \frac{t^3}{3}\Big|_{1}^{4} - \frac{t^2}{2}\Big|_{1}^{4} - 6t\Big|_{1}^{4}$$

$$= \left(\frac{4^3}{3} - \frac{1^3}{3}\right) - \left(\frac{4^2}{2} - \frac{1^2}{2}\right) - (6(4) - 6(1)) = -\frac{9}{2}$$

or

$$\int_{1}^{4} (t^2 - t - 6) dt = \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_{1}^{4} = \left(\frac{4^3}{3} - \frac{4^2}{2} - 6(4)\right) - \left(\frac{1^3}{3} - \frac{1^2}{2} - 6(1)\right) = -\frac{9}{2}$$

(f) $\int_{0}^{1} \sqrt{x} \left(\sqrt[3]{x\sqrt{x}} + \sqrt[4]{x}\right) dx$

Solution: We have

$$\int_{0}^{1} \sqrt{x} \left(\sqrt[3]{x\sqrt{x}} + \sqrt[4]{x}\right) dx = \int_{0}^{1} x^{1/2} \left(\left(x^1 \cdot x^{1/2}\right)^{1/3} + x^{1/4}\right) dx = \int_{0}^{1} x^{1/2} \left(\left(x^{1+1/2}\right)^{1/3} + x^{1/4}\right) dx$$

$$= \int_{0}^{1} x^{1/2} \left(\left(x^{3/2}\right)^{1/3} + x^{1/4}\right) dx = \int_{0}^{1} x^{1/2} \left(x^{3/2 \cdot 1/3} + x^{1/4}\right) dx = \int_{0}^{1} x^{1/2} \left(x^{1/2} + x^{1/4}\right) dx$$

$$= \int_{0}^{1} \left(x^{1/2} \cdot x^{1/2} + x^{1/2} \cdot x^{1/4}\right) dx = \int_{0}^{1} \left(x^{1/2+1/2} + x^{1/2+1/4}\right) dx = \int_{0}^{1} \left(x + x^{3/4}\right) dx$$

$$= \left(\frac{x^2}{2} + \frac{x^{3/4+1}}{3/4+1}\right)\Big|_{0}^{1} = \left(\frac{x^2}{2} + \frac{x^{7/4}}{7/4}\right)\Big|_{0}^{1} = \left(\frac{1}{2}x^2 + \frac{4}{7}x^{7/4}\right)\Big|_{0}^{1}$$

$$= \left(\frac{1}{2} \cdot 1^2 + \frac{4}{7} \cdot 1^{7/4}\right) - \left(\frac{1}{2} \cdot 0^2 + \frac{4}{7} \cdot 0^{7/4}\right) = \frac{1}{2} + \frac{4}{7} = \frac{15}{14}$$

(g) $\int_{-2}^{4} x^2(x^3+8)^2 dx$

Solution 1: By the Fundamental Theorem of Calculus,

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

Since

$$\int x^2(x^3+8)^2 dx = \begin{bmatrix} x^3+8=u \\ d(x^3+8)=du \\ 3x^2dx = du \\ x^2dx = \frac{1}{3}du \end{bmatrix} = \int u^2 \frac{1}{3} du = \frac{1}{3}\int u^2 du = \frac{1}{3}\cdot\frac{u^3}{3} + C = \frac{1}{9}u^3 + C$$

$$= \frac{1}{9}(x^3+8)^3 + C$$

it follows that

$$\int_{-2}^{4} x^2(x^3+8)^2 dx = \frac{1}{9}(x^3+8)^3\Big|_{-2}^{4} = \frac{1}{9}(4^3+8)^3 - \frac{1}{9}((-2)^3+8)^3 = 41,472$$

Solution 2: We have

$$\int_{-2}^{4} x^2(x^3+8)^2 dx = \begin{bmatrix} x^3+8=u \\ d(x^3+8)=du \\ 3x^2dx = du \\ x^2dx = \frac{1}{3}du \end{bmatrix} = \int_{(-2)^3+8}^{4^3+8} u^2\frac{1}{3}du = \frac{1}{3}\int_0^{72} u^2 du = \frac{1}{3}\frac{u^3}{3}\Big|_0^{72} = \frac{1}{9}72^3 = 41,472$$

(h) $\int_0^2 \frac{x}{\sqrt{2x^2+1}}dx$

Solution: We have

$$\int_0^2 \frac{x}{\sqrt{2x^2+1}}dx = \begin{bmatrix} 2x^2+1=u \\ d(2x^2+1)=du \\ 4xdx = du \\ xdx = \frac{1}{4}du \end{bmatrix} = \int_{2\cdot 0^2+1}^{2\cdot 2^2+1} u^{-1/2}\frac{1}{4}du = \frac{1}{4}\int_1^9 u^{-1/2}du = \frac{1}{4}\frac{u^{1/2}}{1/2}\Big|_1^9 = \frac{3}{2} - \frac{1}{2} = 1$$

(i) $\int_0^4 \frac{x}{\sqrt{2x+1}}dx$

Solution: We have

$$\int_0^4 \frac{x}{\sqrt{2x+1}}dx = \begin{bmatrix} 2x+1=u \Rightarrow x = \frac{1}{2}(u-1) \\ d(2x+1)=du \\ 2dx = du \\ dx = \frac{1}{2}du \end{bmatrix} = \int_{2\cdot 0+1}^{2\cdot 4+1} \frac{1}{2}(u-1)u^{-1/2}\frac{1}{2}du = \frac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})du$$

$$= \frac{1}{4}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right)\Big|_1^9 = \frac{10}{3} \approx 3.333$$

Area

If f is a continuous function on $[a, b]$, then

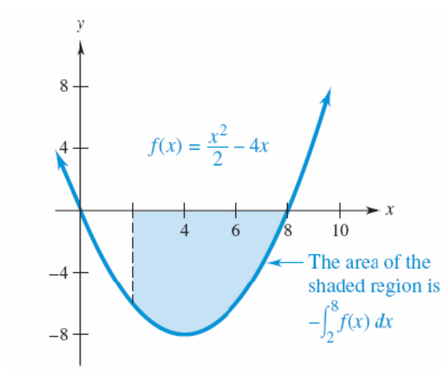
$$\int_a^b f(x)\,dx = \begin{pmatrix} \text{area between the} \\ \text{graph and the } x\text{-axis} \\ \textit{above} \text{ the axis} \end{pmatrix} - \begin{pmatrix} \text{area between the} \\ \text{graph and the } x\text{-axis} \\ \textit{below} \text{ the axis} \end{pmatrix}$$

EXAMPLE: Find the area between the x-axis and the graph

$$f(x) = \frac{1}{2}x^2 - 4x$$

from $x = 2$ to $x = 8$.

Solution: That region, which is shaded in the Figure below, lies below the x-axis.



Therefore

$$A = -\int_2^8 \left(\frac{1}{2}x^2 - 4x\right) dx = -\left(\frac{1}{2}\cdot\frac{x^3}{3} - 4\frac{x^2}{2}\right)\Bigg|_2^8 = -\left(\frac{x^3}{6} - 2x^2\right)\Bigg|_2^8$$

$$= -\left[\left(\frac{8^3}{6} - 2\cdot 8^2\right) - \left(\frac{2^3}{6} - 2\cdot 2^2\right)\right] = 36$$

EXAMPLE: Find the area between the graph of $f(x) = 6x^2 - 7x - 3$ and the x-axis from $x = 0$ to $x = 2$.

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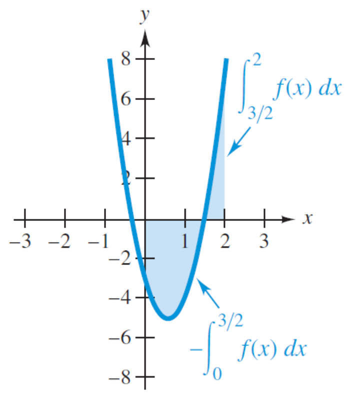
Solution: To find the total area, compute the area under the x-axis and the area over the x-axis separately. Start by finding the x-intercepts of the graph by solving

$$6x^2 - 7x - 3 = 0$$
$$(2x - 3)(3x + 1) = 0$$
$$x = 3/2 \quad \text{or} \quad x = -1/3$$

Since we are concerned just with the graph between 0 and 2, the only relevant x-intercept is 3/2.



$f(x) = 6x^2 - 7x - 3$

We have

$$A = -\int_0^{3/2} f(x)dx + \int_{3/2}^{2} f(x)dx$$

$$= -\int_0^{3/2} (6x^2 - 7x - 3)dx + \int_{3/2}^{2} (6x^2 - 7x - 3)dx$$

$$= -\left(6\frac{x^3}{3} - 7\frac{x^2}{2} - 3x\right)\Bigg|_0^{3/2} + \left(6\frac{x^3}{3} - 7\frac{x^2}{2} - 3x\right)\Bigg|_{3/2}^{2}$$

$$= -\left(2x^3 - \frac{7}{2}x^2 - 3x\right)\Bigg|_0^{3/2} + \left(2x^3 - \frac{7}{2}x^2 - 3x\right)\Bigg|_{3/2}^{2}$$

$$= -\left[\left(2\left(\frac{3}{2}\right)^3 - \frac{7}{2}\left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right)\right) - \left(2(0)^3 - \frac{7}{2}(0)^2 - 3(0)\right)\right]$$

$$+ \left[\left(2(2)^3 - \frac{7}{2}(2)^2 - 3(2)\right) - \left(2\left(\frac{3}{2}\right)^3 - \frac{7}{2}\left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right)\right)\right] = \frac{45}{8} + \frac{13}{8} = \frac{58}{8} = 7.25$$

Applications

EXAMPLE: Data and projections from the U.S. Energy Information Administration were used to generate the following model for the rate of consumption of marketed renewable energy (in trillion BTUs per year), where $x = 10$ corresponds to the beginning of the year 2010:

$$C'(x) = -1.86x^2 + 386x + 4400 \quad (10 \le x \le 35)$$

At this consumption rate, what is the amount of energy that will be used between the beginning of 2010 and the beginning of 2030?

Solution: The amount of marketed renewable energy used is the total change in consumption from year 10 to year 30, so it is given by the definite integral

$$\begin{aligned}
\int_{10}^{30} C'(x)dx &= \int_{10}^{30} (-1.86x^2 + 386x + 4400)dx \\
&= \left(-1.86\frac{x^3}{3} + 386\frac{x^2}{2} + 4400x\right)\Bigg|_{10}^{30} \\
&= \left(-.62x^3 + 193x^2 + 4400x\right)\Big|_{10}^{30} \\
&= \left(-.62(30)^3 + 193(30)^2 + 4400(30)\right) \\
&\quad - \left(-.62(10)^3 + 193(10)^2 + 4400(10)\right) = 226,280
\end{aligned}$$

Therefore, 226,280 trillion BTUs of marketable renewable energy are projected to be used between 2010 and 2030.