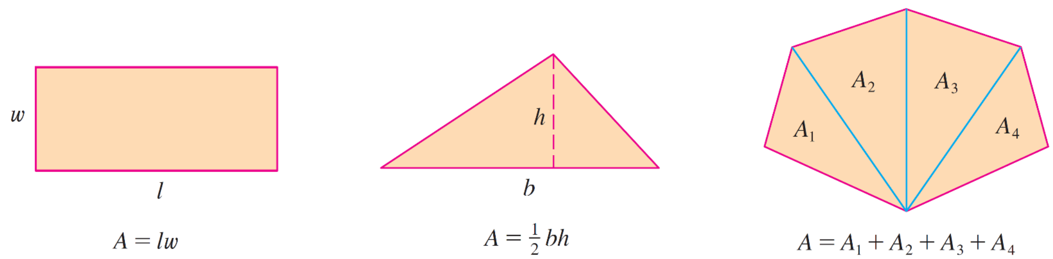
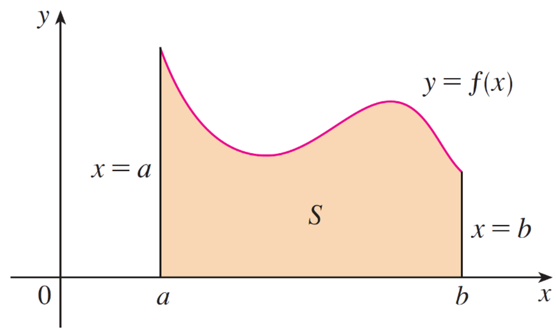
Section 13.3 Area and the Definite Integral

We can easily find areas of certain geometric figures using well-known formulas:



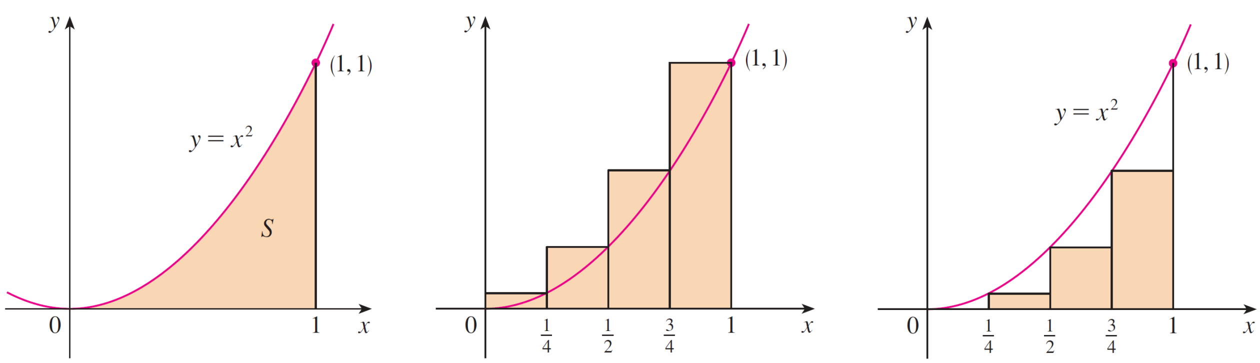
However, it isn't easy to find the area of a region with curved sides:



METHOD: To evaluate the area of the region S we first approximate it by rectangles and then we take the limit of the area of these rectangles as we increase the number of rectangles.

EXAMPLE: Use rectangles to estimate the area under the parabola $f(x) = x^2$ from 0 to 1.

Solution: We first draw pictures:



We have

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2+\frac{1}{4}\cdot\left(\frac{2}{4}\right)^2+\frac{1}{4}\cdot\left(\frac{3}{4}\right)^2+\frac{1}{4}\cdot\left(\frac{4}{4}\right)^2 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2+\frac{1}{4}\cdot\left(\frac{1}{2}\right)^2+\frac{1}{4}\cdot\left(\frac{3}{4}\right)^2+\frac{1}{4}\cdot 1^2 = \frac{15}{32} = 0.46875$$

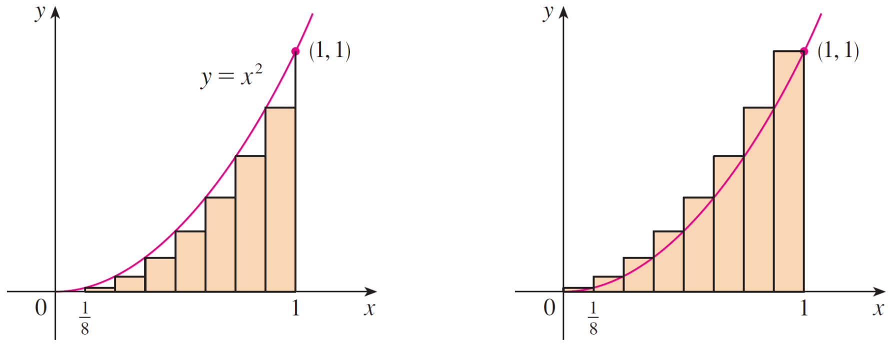
and

$$L_4 = \frac{1}{4}\cdot 0^2+\frac{1}{4}\cdot\left(\frac{1}{4}\right)^2+\frac{1}{4}\cdot\left(\frac{2}{4}\right)^2+\frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{1}{4}\cdot 0^2+\frac{1}{4}\cdot\left(\frac{1}{4}\right)^2+\frac{1}{4}\cdot\left(\frac{1}{2}\right)^2+\frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

Thus

$$0.21875 < A < 0.46875$$

If we repeat this procedure with 8 strips, we get a better approximation:

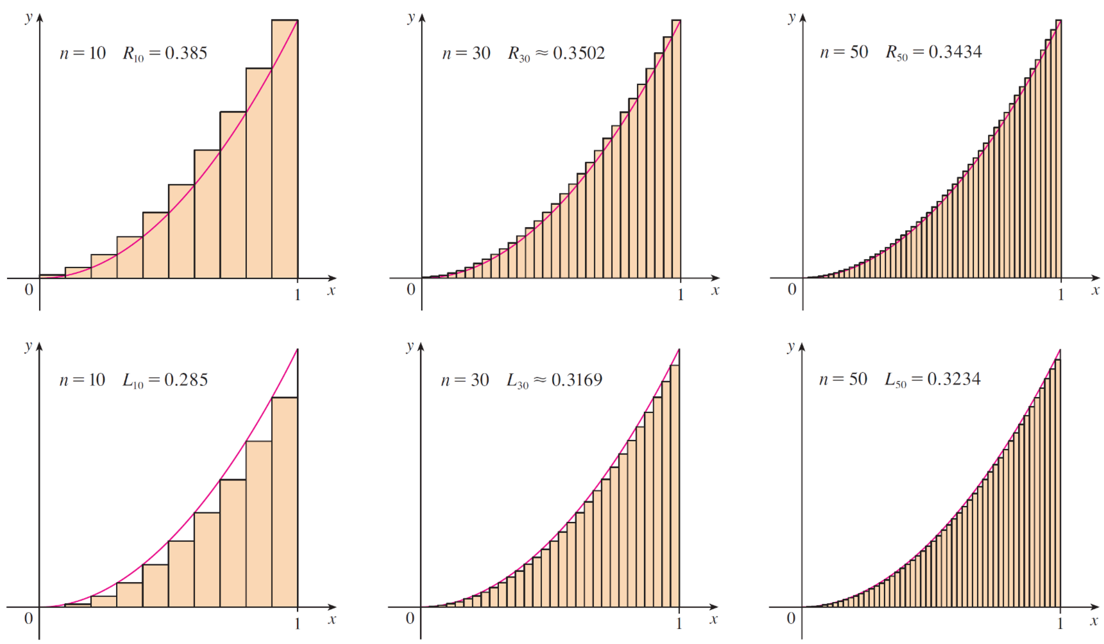


(a) Using left endpoints (b) Using right endpoints

and

$$0.2734375 < A < 0.3984375$$

More strips we use, better approximation we get:



We summarize this in the following table:

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

REMARK: One can show that the area under the parabola $f(x) = x^2$ from 0 to 1 is 1/3 (see Appendix).

EXAMPLE: Use four rectangles with (a) right endpoints, (b) left endpoints, (c) middle points to estimate the area under the parabola $y = x^2 + 1$ from 0 to 2.

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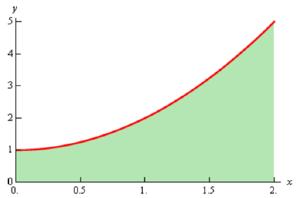
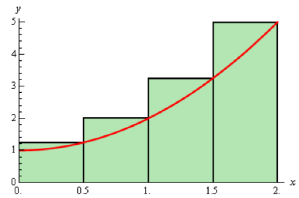
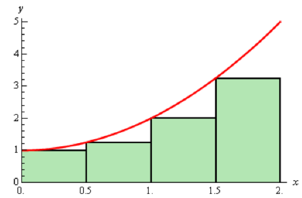
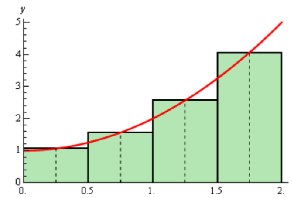
Solution: We first draw pictures:

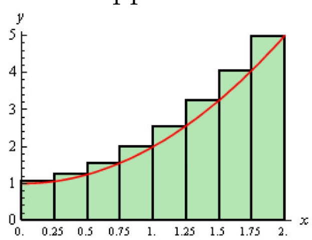
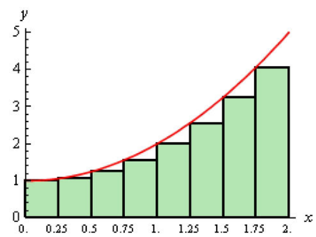
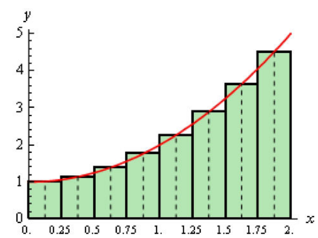
We have

$$R_4 = \frac{1}{2}\cdot f\left(\frac{1}{2}\right) + \frac{1}{2}\cdot f(1) + \frac{1}{2}\cdot f\left(\frac{3}{2}\right) + \frac{1}{2}\cdot f(2) = \frac{1}{2}\cdot\frac{5}{4} + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{13}{4} + \frac{1}{2}\cdot 5 = 5.75$$

$$L_4 = \frac{1}{2}\cdot f(0) + \frac{1}{2}\cdot f\left(\frac{1}{2}\right) + \frac{1}{2}\cdot f(1) + \frac{1}{2}\cdot f\left(\frac{3}{2}\right) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{5}{4} + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{13}{4} = 3.75$$

$$M_4 = \frac{1}{2}\cdot f\left(\frac{1}{4}\right) + \frac{1}{2}\cdot f\left(\frac{3}{4}\right) + \frac{1}{2}\cdot f\left(\frac{5}{4}\right) + \frac{1}{2}\cdot f\left(\frac{7}{4}\right) = \frac{1}{2}\cdot\frac{17}{16} + \frac{1}{2}\cdot\frac{25}{16} + \frac{1}{2}\cdot\frac{41}{16} + \frac{1}{2}\cdot\frac{65}{16} = 4.625$$

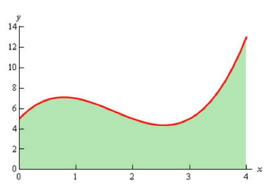
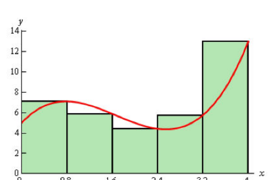
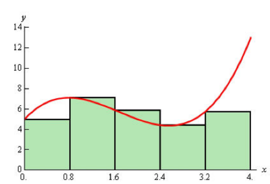
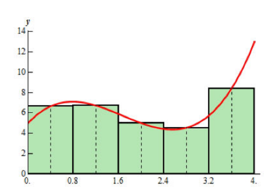
For comparison's sake the exact area is $A = \dfrac{14}{3} = 4.\overline{6}$. If we repeat this procedure with 8 strips, we get a better approximation:

and $R_8 = 5.1875$, $L_8 = 4.1875$, $M_8 = 4.65625$.

EXAMPLE: Use five rectangles with (a) right end points, (b) left endpoints, (c) middle points to estimate the area under the parabola $y = x^3 - 5x^2 + 6x + 5$ from 0 to 4.

Solution: We first draw pictures:

We have

$$R_5 = \frac{4}{5}\cdot f\left(\frac{4}{5}\right) + \frac{4}{5}\cdot f\left(\frac{8}{5}\right) + \frac{4}{5}\cdot f\left(\frac{12}{5}\right) + \frac{4}{5}\cdot f\left(\frac{16}{5}\right) + \frac{4}{5}\cdot f\left(\frac{20}{5}\right) = 28.96$$

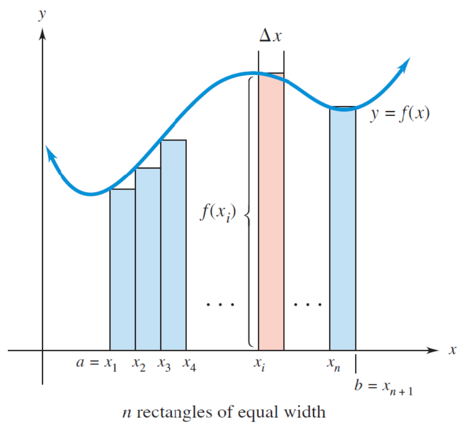
$$L_5 = \frac{4}{5}\cdot f(0) + \frac{4}{5}\cdot f\left(\frac{4}{5}\right) + \frac{4}{5}\cdot f\left(\frac{8}{5}\right) + \frac{4}{5}\cdot f\left(\frac{12}{5}\right) + \frac{4}{5}\cdot f\left(\frac{16}{5}\right) = 22.56$$

$$\begin{aligned} M_5 &= \frac{4}{5}\cdot f\left(\frac{2}{5}\right) + \frac{4}{5}\cdot f\left(\frac{2}{5}+\frac{4}{5}\right) + \frac{4}{5}\cdot f\left(\frac{2}{5}+\frac{8}{5}\right) + \frac{4}{5}\cdot f\left(\frac{2}{5}+\frac{12}{5}\right) + \frac{4}{5}\cdot f\left(\frac{2}{5}+\frac{16}{5}\right) \\ &= \frac{4}{5}\cdot f\left(\frac{2}{5}\right) + \frac{4}{5}\cdot f\left(\frac{6}{5}\right) + \frac{4}{5}\cdot f\left(\frac{10}{5}\right) + \frac{4}{5}\cdot f\left(\frac{14}{5}\right) + \frac{4}{5}\cdot f\left(\frac{18}{5}\right) = 25.12 \end{aligned}$$

For comparison's sake the exact area is $A = \dfrac{76}{3} = 25.\overline{3}$.

DEFINITION: The **area** A of the region S that lies under the graph of the continuous non-negative function f is the limit of the sum of the areas of approximating rectangles:

$$\boxed{A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]}$$



REMARK: It can also be shown that we get the <u>same</u> value if we use right endpoints

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_{n+1})\Delta x]$$

DEFINITION OF A DEFINITE INTEGRAL: If f is a continuous function defined on $[a, b]$, the **definite integral** of f from a to b is a number

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]}$$

EXAMPLE: Approximate $\int_0^1 x^2dx$, which is the area under the graph of $f(x) = x^2$, above the x-axis, and between $x = 0$ and $x = 1$, by using four rectangles of equal width whose heights are the values of the function at the left endpoint of each rectangle.

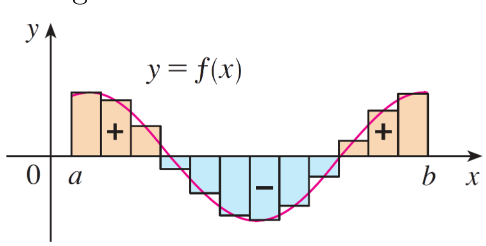
Solution: In Example 1 it was shown that

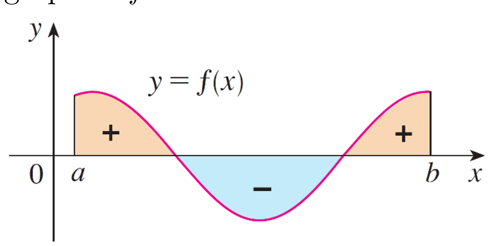
$$\int_0^1 x^2dx \approx \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

REMARK: Note that the definition of the definite integral is valid even when $f(x)$ takes negative values. A definite integral can be interpreted as a **net area,** that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.





Applications

<u>THE DISTANCE PROBLEM</u>: Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

EXAMPLE: Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time(s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time(s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant. If we take the velocity during that time interval to be the initial velocity (25 ft/s), then we obtain the approximate distance traveled during the first five seconds

$$25 \text{ ft/s} \times 5 \text{ s} = 125 \text{ ft}$$

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when $t = 5$ s. So, our estimate for the distance traveled from $t = 5$ s to $t = 10$ s is

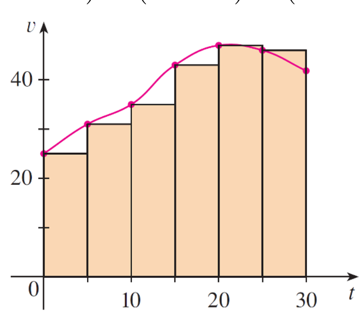
$$31 \text{ ft/s} \times 5 \text{ s} = 155 \text{ ft}$$

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

$$(25 \times 5) + (31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5) = 1135 \text{ ft}$$

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity. Then our estimate becomes

$$(31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5) + (41 \times 5) = 1215 \text{ ft}$$



In general, the total distance traveled during the time interval $[a, b]$ is approximately

$$v(t_1)\Delta t + v(t_2)\Delta t + \ldots + v(t_n)\Delta t \quad \text{where} \quad \Delta t = \frac{b-a}{n}$$

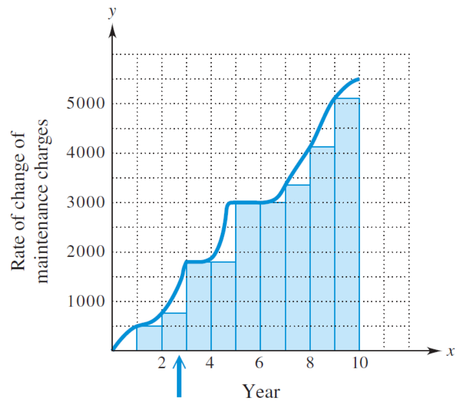
The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the <u>*exact*</u> distance d traveled is the <u>*limit*</u> of the expression above:

$$d = \lim_{n\to\infty} [v(t_1)\Delta t + v(t_2)\Delta t + \ldots + v(t_n)\Delta t] = \int_a^b v(x)dx = s(b) - s(a)$$

Total Change in $F(x)$

Let f be a function such that f is continuous on the interval $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$. If $f(x)$ is the rate of change of a function $F(x)$, then the **total change in $F(x)$** as x goes from a to b is the area between the graph of $f(x)$ and the x-axis from $x = a$ to $x = b$.

EXAMPLE: The Figure below shows the graph of the function that gives the rate of change of the annual maintenance charges for a certain machine. The rate function is increasing because maintenance tends to cost more as the machine gets older. Estimate the total maintenance charges over the 10-year life of the machine.



Solution: This is the situation described in the preceding box, with $F(x)$ being the maintenance cost function and $f(x)$, whose graph is given, the rate-of-change function. The total maintenance charges are the total change in $F(x)$ from $x = 0$ to $x = 10$ — that is, the area between the graph of the rate function and the x-axis from $x = 0$ to $x = 10$. We can approximate this area by using the shaded rectangles in the Figure above. For instance, the rectangle marked with an arrow has base 1 (from year 2 to year 3) and height 750 (the rate of change at $x = 2$), so its area is $1 \times 750 = 750$. Similarly, each of the other rectangles has base 1 and height determined by the rate of change at the beginning of the year. Consequently, we estimate the area to be the sum

$$1 \cdot 0 + 1 \cdot 500 + 1 \cdot 750 + 1 \cdot 1800 + 1 \cdot 1800 + 1 \cdot 3000 + 1 \cdot 3000 + 1 \cdot 3400 + 1 \cdot 4200 + 1 \cdot 5200 = 23,650$$

Hence, the total maintenance charges over the 10 years are at least \$23,650. (The unshaded areas under the rate graph have not been accounted for in this estimate).

EXAMPLE: The marginal cost (in dollars per snowboard) for manufacturing x snowboards is given by

$$MC(x) = \frac{(x-600)^2}{450}$$

Find the amount added to the total cost when production goes from 350 to 400 snow boards.

Solution: The amount added is the total change in cost from $x = 350$ to $x = 400$. Since the marginal cost function is the derivative of the cost function,

$$\text{Total change} = \int\limits_{350}^{400} \frac{(x-600)^2}{450} dx$$

Numerical integration shows that the total change in cost from $x = 350$ to $x = 400$ is \$5648.15.

Appendix

EXAMPLE: For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is

$$\lim_{n\to\infty} R_n = \frac{1}{3}$$

Solution: We have

$$\begin{aligned}
R_n &= \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{3}{n}\right)^2 + \ldots + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^2 \\
&= \frac{1}{n}\cdot\frac{1^2}{n^2} + \frac{1}{n}\cdot\frac{2^2}{n^2} + \frac{1}{n}\cdot\frac{3^2}{n^2} + \ldots + \frac{1}{n}\cdot\frac{n^2}{n^2} \\
&= \frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \ldots + \frac{n^2}{n^3} \\
&= \frac{1}{n^3}\cdot 1^2 + \frac{1}{n^3}\cdot 2^2 + \frac{1}{n^3}\cdot 3^2 + \ldots + \frac{1}{n^3}\cdot n^2 \\
&= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)
\end{aligned}$$

It is known that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

therefore

$$R_n = \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Thus we have

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2} \\
&= \lim_{n\to\infty}\left(\frac{2n^2}{6n^2} + \frac{3n}{6n^2} + \frac{1}{6n^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) \\
&= \frac{1}{3} + 0 + 0 = \frac{1}{3}
\end{aligned}$$

or

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty}\frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n\to\infty}\frac{1}{6}\left(\frac{n}{n} + \frac{1}{n}\right)\left(\frac{2n}{n} + \frac{1}{n}\right) \\
&= \lim_{n\to\infty}\frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\
&= \frac{1}{6}\cdot(1+0)\cdot(2+0) = \frac{1}{3}
\end{aligned}$$