Section 13.1 Antiderivatives

DEFINITION: A function F is called an **antiderivative** of f on an open interval I if

$$F'(x) = f(x) \text{ for all } x \text{ in } I$$

EXAMPLES:

1. If $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$, since

$$F'(x) = \left(\frac{1}{3}x^3\right)' = \frac{1}{3}\left(x^3\right)' = \frac{1}{3} \cdot 3x^2 = x^2$$

2. If $f(x) = x$, then $F(x) = \frac{1}{2}x^2$, since

$$F'(x) = \left(\frac{1}{2}x^2\right)' = \frac{1}{2}\left(x^2\right)' = \frac{1}{2} \cdot 2x = x$$

3. If $f(x) = x^4$, then $F(x) = \frac{1}{5}x^5$, since

$$F'(x) = \left(\frac{1}{5}x^5\right)' = \frac{1}{5}\left(x^5\right)' = \frac{1}{5} \cdot 5x^4 = x^4$$

IN GENERAL:

$$\boxed{\text{If } f(x) = x^n, \text{ then } F(x) = \frac{x^{n+1}}{n+1}, \qquad n \neq -1}$$

REMARK: Note that if $f(x) = x^n$ and $F(x) = \frac{x^{n+1}}{n+1}$, then $F(x) + C$ is again an antiderivative of $f(x)$, since

$$(F(x) + C)' = \left(\frac{x^{n+1}}{n+1} + C\right)' = \left(\frac{x^{n+1}}{n+1}\right)' + C' = x^n + 0 = x^n$$

THEOREM: If F is an antiderivative of f on an open interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

NOTATION: To denote the set of all antiderivatives of f on an open interval I we use the **indefinite integral** notation:

$$\boxed{\int f(x)dx = F(x) + C}$$

Table of Basic Indefinite Integrals

$\int cf(x)dx = c\int f(x)dx$	$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
$\int cdx = cx + C$	$\int xdx = \frac{x^2}{2} + C$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\int e^x dx = e^x + C$	$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

EXAMPLES:

1. $\int dx = \int 1dx = x + C, \qquad \int 5dx = 5x + C, \qquad \int \frac{3}{4}dx = \frac{3}{4}x + C$

2. $\int xdx = \int x^1 dx = [PR\ with\ n = 1] = \frac{x^{1+1}}{1+1} + C = \frac{x^2}{2} + C$

3. $\int x^{-8}dx = [PR\ with\ n = -8] = \frac{x^{-8+1}}{-8+1} + C = \frac{x^{-7}}{-7} + C = -\frac{1}{7}x^{-7} + C$

4. $\int \sqrt{x}dx = \int x^{1/2}dx = [PR\ with\ n = 1/2] = \frac{x^{1/2+1}}{1/2+1} + C = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$

5. $\int \frac{1}{\sqrt[5]{x}}dx = \int \frac{1}{x^{1/5}}dx = \int x^{-1/5}dx = [PR\ with\ n = -1/5] = \frac{x^{-1/5+1}}{-1/5+1} + C = \frac{5}{4}x^{4/5} + C$

6. $\int x\sqrt[3]{x}dx = \int x^1 \cdot x^{1/3}dx = \int \underbrace{x^{1+1/3}}_{x^{4/3}}dx = [PR\ with\ n = 4/3] = \frac{x^{4/3+1}}{4/3+1} + C = \frac{3}{7}x^{7/3} + C$

7. $\int \frac{x^4}{\sqrt[5]{x^3}}dx = \int \frac{x^4}{x^{3/5}}dx = \int \underbrace{x^{4-3/5}}_{x^{17/5}}dx = [PR\ with\ n = 17/5] = \frac{x^{17/5+1}}{17/5+1} + C = \frac{5}{22}x^{22/5} + C$

8. $\int \frac{x\sqrt[3]{x}}{\sqrt{x}}dx$

9. $\int x(1 + x^2)dx$

10. $\int x(1 + x^2)^2 dx$

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$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$	$\int \frac{1}{x}dx = \ln\|x\| + C$
$\int e^x dx = e^x + C$	$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

8. $$\int \frac{x\sqrt[3]{x}}{\sqrt{x}}dx = \int \frac{x^1 \cdot x^{1/3}}{x^{1/2}}dx = \int \frac{x^{1+1/3}}{x^{1/2}}dx$$
$$= \int \underbrace{x^{1+1/3-1/2}}_{x^{5/6}}dx = [PR \ with \ n = 5/6] = \frac{x^{5/6+1}}{5/6+1} + C$$
$$= \frac{6}{11}x^{11/6} + C$$

9. $$\int x(1+x^2)dx = \int \left(x + x^3\right)dx = \int xdx + \int x^3 dx = \frac{x^2}{2} + \frac{x^4}{4} + C$$

10. $$\int x(1+x^2)^2 dx = \int x\left(1 + 2x^2 + x^4\right)dx$$
$$= \int (x + 2x^3 + x^5)dx$$
$$= \int xdx + 2\int x^3dx + \int x^5dx$$
$$= \frac{x^2}{2} + 2\frac{x^4}{4} + \frac{x^6}{6} + C$$
$$= \frac{1}{2}x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + C$$

Table of Basic Indefinite Integrals

$\int cf(x)dx = c\int f(x)dx$	$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
$\int cdx = cx + C$	$\int xdx = \frac{x^2}{2} + C$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\int e^x dx = e^x + C$	$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

11.
$$\begin{aligned}
\int \left(\frac{11x^2 - 5x^{-1/2} + 4x - 2}{7\sqrt{x}}\right) dx &= \int \left(\frac{11x^2 - 5x^{-1/2} + 4x - 2}{7x^{1/2}}\right) dx \\
&= \int \left(\frac{11x^2}{7x^{1/2}} - \frac{5x^{-1/2}}{7x^{1/2}} + \frac{4x}{7x^{1/2}} - \frac{2}{7x^{1/2}}\right) dx \\
&= \int \left(\frac{11}{7}x^{2-1/2} - \frac{5}{7}x^{-1/2-1/2} + \frac{4}{7}x^{1-1/2} - \frac{2}{7}x^{-1/2}\right) dx \\
&= \int \left(\frac{11}{7}x^{3/2} - \frac{5}{7}x^{-1} + \frac{4}{7}x^{1/2} - \frac{2}{7}x^{-1/2}\right) dx \\
&= \frac{11}{7}\int x^{3/2}dx - \frac{5}{7}\int x^{-1}dx + \frac{4}{7}\int x^{1/2}dx - \frac{2}{7}\int x^{-1/2}dx \\
&= \frac{11}{7}\cdot\frac{x^{3/2+1}}{3/2+1} - \frac{5}{7}\ln|x| + \frac{4}{7}\cdot\frac{x^{1/2+1}}{1/2+1} - \frac{2}{7}\cdot\frac{x^{-1/2+1}}{-1/2+1} + C \\
&= \frac{11}{7}\cdot\frac{x^{5/2}}{5/2} - \frac{5}{7}\ln|x| + \frac{4}{7}\cdot\frac{x^{3/2}}{3/2} - \frac{2}{7}\cdot\frac{x^{1/2}}{1/2} + C \\
&= \frac{11}{7}\cdot\frac{2}{5}x^{5/2} - \frac{5}{7}\ln|x| + \frac{4}{7}\cdot\frac{2}{3}x^{3/2} - \frac{2}{7}\cdot 2x^{1/2} + C \\
&= \frac{22}{35}x^{5/2} - \frac{5}{7}\ln|x| + \frac{8}{21}x^{3/2} - \frac{4}{7}x^{1/2} + C
\end{aligned}$$

12.
$$\int \left(9x^2 + 4e^x\right) dx = 9\int x^2 dx + 4\int e^x dx = 9\cdot\frac{x^3}{3} + 4e^x + C = 3x^3 + 4e^x + C$$

13.
$$\begin{aligned}
\int \left(7e^{2x} + \frac{e^{x/3}}{5}\right) dx = \int 7e^{2x}dx + \int \frac{1}{5}e^{x/3}dx &= 7\int e^{2x}dx + \frac{1}{5}\int e^{x/3}dx \\
&= 7\cdot\frac{1}{2}e^{2x} + \frac{1}{5}\cdot\frac{1}{1/3}e^{x/3} + C \\
&= \frac{7}{2}e^{2x} + \frac{1}{5}\cdot\frac{3}{1}e^{x/3} + C \\
&= \frac{7}{2}e^{2x} + \frac{3}{5}e^{x/3} + C
\end{aligned}$$

Applications

EXAMPLE: The rate of increase in the number of cell phone subscriptions worldwide (in millions per year) can be approximated by

$$S'(x) = 56x + 166 \qquad (*)$$

where $x = 0$ corresponds to the year 2000. There were 5960 million cell phone subscriptions worldwide in 2011. Find the function $S(x)$ that gives the number of subscribers (in millions) in year x. (Data from: www.worldbank.org.)

Solution: To find the number of subscribers $S(x)$, we integrate both sides of $(*)$:

$$S'(x) = 56x + 166$$

$$\begin{aligned} S(x) = \int S'(x)dx &= \int (56x + 166)dx \\ &= \int 56x dx + \int 166 dx \\ &= 56 \int x dx + \int 166 dx \\ &= 56 \cdot \frac{x^2}{2} + 166x + C \\ &= 28x^2 + 166x + C \end{aligned}$$

To find the value of C, use the fact that in 2011 ($x = 11$) there were 5960 million subscribers, which says that $S(11) = 5960$. This information gives an equation that can be solved for C:

$$\begin{aligned} S(x) &= 28x^2 + 166x + C \\ S(11) &= 28(11)^2 + 166(11) + C \\ 5960 &= 28(11)^2 + 166(11) + C \\ 5960 &= 5214 + C \\ C &= 5960 - 5214 = 746 \end{aligned}$$

Thus, the number of subscribers in year x is

$$S(x) = 28x^2 + 166x + 746$$

EXAMPLE: Suppose the marginal revenue from selling x units of a product is given by $40/e^{.05x}+10$.

(a) Find the revenue function.

Solution: The marginal revenue is the derivative of the revenue function, so

$$\frac{dR}{dx} = \frac{40}{e^{.05x}} + 10$$

$$R = \int\left(\frac{40}{e^{.05x}}+10\right)dx = \int\left(40e^{-.05x}+10\right)dx = \int 40e^{-.05x}dx + \int 10dx$$

$$= 40\int e^{-.05x}dx + \int 10dx$$

$$= 40\cdot\frac{1}{-.05}e^{-.05x} + 10x + C$$

$$= -800e^{-.05x} + 10x + C$$

where C is a constant. If $x=0$, then $R=0$ (no items sold means no revenue), and

$$0 = -800e^{0} + 10(0) + C$$

$$0 = -800\cdot 1 + 0 + C$$

$$0 = -800 + C$$

$$800 = C$$

Thus,

$$R = -800e^{-.05x} + 10x + 800$$

is the revenue function.

(b) Find the demand function for this product.

Solution: Recall that $R = xp$, where p is the demand function. Hence,

$$-800e^{-.05x} + 10x + 800 = xp$$

$$\frac{-800e^{-.05x} + 10x + 800}{x} = p$$

The demand function is $p = \dfrac{-800e^{-.05x} + 10x + 800}{x}$.