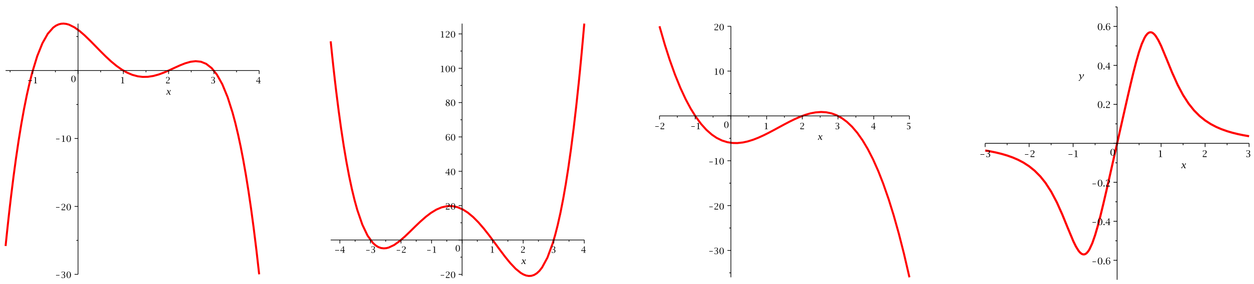
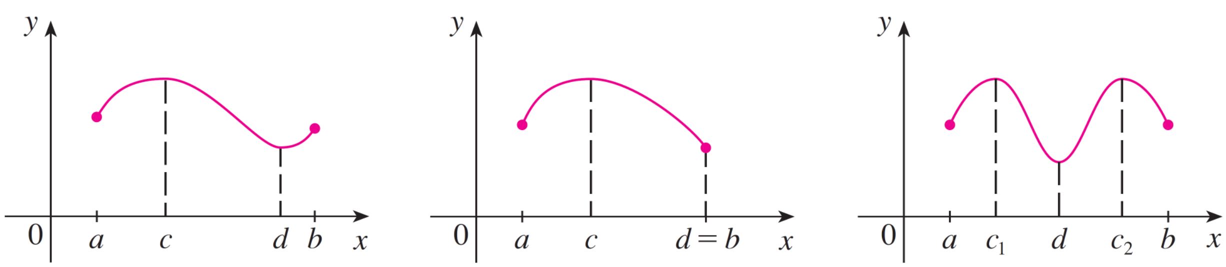
Section 12.3 Optimization Applications

DEFINITION: A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** (or **global minimum**) at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.



DEFINITION: A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c. [This means that $f(c) \geq f(x)$ for all x in some open interval containing c.] Similarly, f has a **local minimum** (or **relative minimum**) at c if $f(c) \leq f(x)$ when x is near c.

THEOREM (The Extreme Value Theorem): If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



THE CLOSED INTERVAL METHOD: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).

2. Find the values of f at the endpoints of the interval.

3. The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest value of these values is the absolute minimum value.

EXAMPLE: Find the absolute maximum and minimum values of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$ and determine where these values occur.

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Solution:

Step 1: Since

$$\begin{aligned} f'(x) = (2x^3 - 15x^2 + 36x)' &= (2x^3)' - (15x^2)' + (36x)' \\ &= 2(x^3)' - 15(x^2)' + 36(x)' \\ &= 2(3x^2) - 15(2x) + 36(1) \\ &= 6x^2 - 30x + 36 \\ &= 6(x^2 - 5x + 6) \\ &= 6(x - 2)(x - 3) \end{aligned}$$

there are two critical numbers $x = 2$ and $x = 3$.

Step 2: We now evaluate f at these critical numbers and at the endpoints $x = 1$ and $x = 5$. We have

$$f(1) = 23, \quad f(2) = 28, \quad f(3) = 27, \quad f(5) = 55$$

Step 3: The largest value is 55 and the smallest value is 23. Therefore the absolute maximum of f on $[1, 5]$ is 55, occurring at $x = 5$ and the absolute minimum of f on $[1, 5]$ is 23, occurring at $x = 1$.

EXAMPLE: Find the absolute maximum and minimum values of $f(x) = 2x^3 - 15x^2 + 24x + 2$ on $[0, 2]$ and determine where these values occur.

Solution:

Step 1: Since

$$\begin{aligned} f'(x) = (2x^3 - 15x^2 + 24x + 2)' &= (2x^3)' - (15x^2)' + (24x)' + (2)' \\ &= 2(x^3)' - 15(x^2)' + 24(x)' + (2)' \\ &= 2(3x^2) - 15(2x) + 24(1) + 0 \\ &= 6x^2 - 30x + 24 \\ &= 6(x^2 - 5x + 4) \\ &= 6(x - 4)(x - 1) \end{aligned}$$

there are two critical numbers $x = 1$ and $x = 4$.

Step 2: Since $x = 4$ is not from $[0, 2]$, we evaluate f only at $x = 1$ and at the endpoints $x = 0$ and $x = 2$. We have

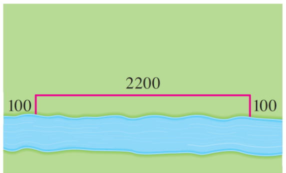
$$f(0) = 2, \quad f(1) = 13, \quad f(2) = 6$$

Step 3: The largest value is 13 and the smallest value is 2. Therefore the absolute maximum of f on $[0, 2]$ is 13, occurring at $x = 1$ and the absolute minimum of f on $[0, 2]$ is 2, occurring at $x = 0$.

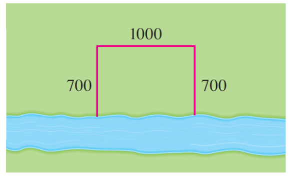
Applications

EXAMPLE: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

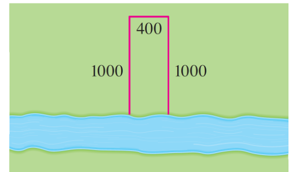
Solution: Note that the area of the field depends on its dimensions:



Area = 100 · 2200 = 220,000 ft^2

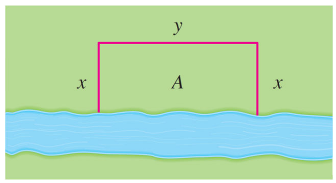


Area = 700 · 1000 = 700,000 ft^2



Area = 1000 · 400 = 400,000 ft^2

To solve the problem, we first draw a picture that illustrates the general case:



The next step is to create a corresponding mathematical model:

$$\text{Maximize: } A = xy$$
$$\text{Constraint: } 2x + y = 2400$$

We now solve the second equation for y and substitute the result into the first equation to express A as a function of one variable:

$$2x + y = 2400 \quad \Longrightarrow \quad y = 2400 - 2x \quad \Longrightarrow \quad A = xy = x(2400 - 2x) = 2400x - 2x^2$$

To find the absolute maximum value of $A = 2400x - 2x^2$, we use

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1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest value of these values is the absolute minimum value.

We first show that $0 \le x \le 1200$. Indeed,

$$y \ge 0 \quad \Longrightarrow \quad 2400 - 2x \ge 0 \quad \Longrightarrow \quad 2400 \ge 2x \quad \Longrightarrow \quad 1200 \ge x$$

On the other hand, $x \ge 0$. Combining these two inequalities gives $0 \le x \le 1200$. The derivative of $A(x)$ is

$$A'(x) = (2400x - 2x^2)' = 2400x' - 2(x^2)' = 2400 \cdot 1 - 2 \cdot 2x = 2400 - 4x$$

so to find the critical numbers we solve the equation

$$2400 - 4x = 0 \quad \Longrightarrow \quad 2400 = 4x \quad \Longrightarrow \quad x = \frac{2400}{4} = 600$$

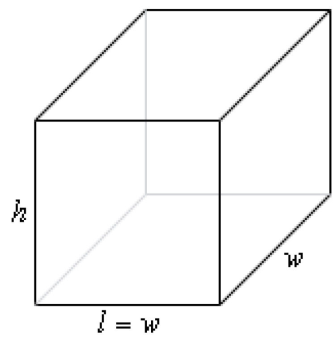
To find the maximum value of $A(x)$ we evaluate it at the end points and critical number:

$$A(0) = 0, \qquad A(600) = 2400 \cdot 600 - 2 \cdot 600^2 = 720,000, \qquad A(1200) = 0$$

The Closed Interval Method gives the maximum value as $A(600) = 720,000$ ft^2 and the dimensions are $x = 600$ ft, $y = 2400 - 2 \cdot 600 = 1200$ ft.

EXAMPLE: We want to construct a box with a square base and we only have 10 m^2 of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.

Solution: We first draw a picture:



The next step is to create a corresponding mathematical model:

Maximize: $V = lwh = w^2h$

Constraint: $2lw + 2wh + 2lh = 2w^2 + 2wh + 2wh = 2w^2 + 4wh = 10$

We now solve the second equation for h and substitute the result into the first equation to express V as a function of one variable:

$$2w^2 + 4wh = 10 \implies h = \frac{10 - 2w^2}{4w} = \frac{5 - w^2}{2w} \implies V = w^2h = w^2\left(\frac{5 - w^2}{2w}\right) = \frac{1}{2}(5w - w^3)$$

Note that we can't use the Closed Interval Method because the domain of $V(w)$ is $(0, \infty)$ which is not a finite interval. Instead, we will use

CRITICAL-POINT THEOREM: Suppose that a function f is continuous on an interval I and that f has exactly one critical number in the interval I, say, $x = c$.

(a)) If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then $f(c)$ is the absolute maximum value of f on the interval I.

(b) If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then $f(c)$ is the absolute minimum value of f on the interval I.

The derivative of $V(w)$ is

$$V'(w) = \left(\frac{1}{2}(5w - w^3)\right)' = \frac{1}{2}\left(5w - w^3\right)' = \frac{1}{2}(5 - 3w^2)$$

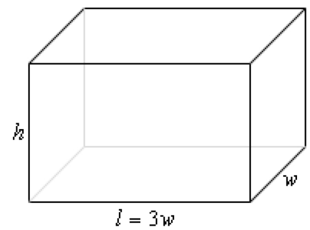
Since $w > 0$, the only critical number is $w = \sqrt{\frac{5}{3}}$. It is easy to see that $V'(w) > 0$ for $0 < w < \sqrt{\frac{5}{3}}$ and $V'(w) < 0$ for $w > \sqrt{\frac{5}{3}}$. Therefore the maximum value of the volume must occur at $w = \sqrt{\frac{5}{3}}$. Finally, the dimensions of the box are

$$w = l = \sqrt{\frac{5}{3}} \approx 1.2910 \text{ m}, \quad h = \frac{5 - w^2}{2w} \approx 1.2910 \text{ m}$$

which means the box with the maximum volume $V = \left(\sqrt{\frac{5}{3}}\right)^3 \approx 2.1517 \text{ m}^3$ is a cube.

EXAMPLE: We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50 ft^3 determine the dimensions that will minimize the cost to build the box.

Solution: We first draw a picture:



The next step is to create a corresponding mathematical model:

Minimize: $C = 10(2lw) + 6(2wh + 2lh) = 10(2 \cdot 3w \cdot w) + 6(2wh + 2 \cdot 3w \cdot h) = 60w^2 + 48wh$

Constraint: $V = lwh = 3w \cdot wh = 3w^2h = 50$

We now solve the second equation for h and substitute the result into the first equation to express C as a function of one variable:

$$3w^2h = 50 \quad \Longrightarrow \quad h = \frac{50}{3w^2} \quad \Longrightarrow \quad C = 60w^2 + 48wh = 60w^2 + 48w \cdot \frac{50}{3w^2} = 60w^2 + \frac{800}{w}$$

Since $w > 0$, we can't use the Closed Interval Method. However, the Critical-Point Theorem can be applied. Indeed, the derivative of $C(w)$ is

$$C'(w) = \left(60w^2 + \frac{800}{w}\right)' = \left(60w^2 + 800w^{-1}\right)' = 60(w^2)' + 800(w^{-1})'$$

$$= 60 \cdot 2w + 800 \cdot (-1)w^{-1-1} = 120w - 800w^{-2} = \frac{120w}{1} - \frac{800}{w^2}$$

$$= \frac{120w^3}{w^2} - \frac{800}{w^2} = \frac{120w^3 - 800}{w^2}$$

The critical numbers occur at the numbers in the domain of the cost function where the value of the derivative $C'(w)$ is zero or does not exist. Since $C'(w)$ always exists for $w > 0$, the only critical numbers occur where $C'(w)$, that is where the numerator is zero. Therefore we can use the Critical-Point Theorem. We have

$$120w^3 - 800 = 0$$

$$120w^3 = 800$$

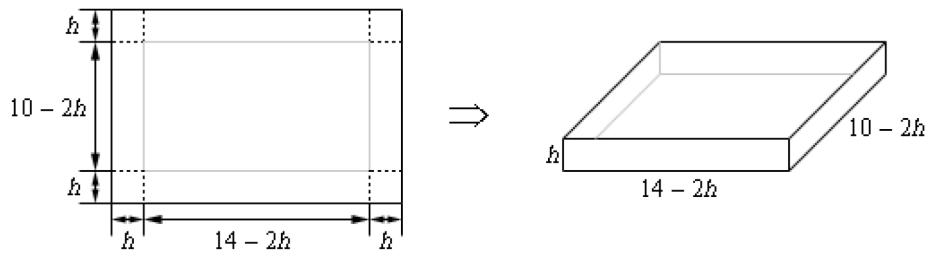
$$w^3 = \frac{800}{120} = \frac{20}{3} \quad \Longrightarrow \quad w = \sqrt[3]{\frac{20}{3}} \approx 1.8821$$

It is easy to see that $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{20}{3}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{20}{3}}$. Therefore the minimum value of the cost must occur at $w = \sqrt[3]{\frac{20}{3}}$. The dimensions are

$$w = \sqrt[3]{\frac{20}{3}} \approx 1.8821 \text{ ft}, \quad l = 3w = 3\sqrt[3]{\frac{20}{3}} \approx 5.6463 \text{ ft}, \quad h = \frac{50}{3w^2} \approx 4.7050 \text{ ft}$$

and the minimum cost is $C\left(\sqrt[3]{\frac{20}{3}}\right) \approx \$637.60$.

EXAMPLE: We have a piece of cardboard that is 14 in by 10 in and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.



Solution: We create a corresponding mathematical model:

$$\text{Maximize: } V = h(14-2h)(10-2h) = 140h - 48h^2 + 4h^3$$

It is easy to see that $0 \le h \le 5$. Therefore we can use either the Closed Interval Method or the Critical-Point Theorem to find the absolute maximum value of $V = 140h - 48h^2 + 4h^3$.

Closed Interval Method: The derivative of $V(h)$ is

$$\begin{aligned} V'(h) = (140h - 48h^2 + 4h^3)' &= (140h)' - (48h^2)' + (4h^3)' \\ &= 140(h)' - 48(h^2)' + 4(h^3)' \\ &= 140(1) - 48(2h) + 4(3h^2) \\ &= 140 - 96h + 12h^2 \end{aligned}$$

so to find the critical numbers we solve the equation

$$140-96h+12h^2 = 0 \implies h = \frac{-(-96) \pm \sqrt{(-96)^2 - 4\cdot 12\cdot 140}}{2\cdot 12} = \frac{12 \pm \sqrt{39}}{3} \approx 1.9183,\ 6.0817$$

Since $0 \le h \le 5$, the only critical number that we must consider is $h = \frac{12-\sqrt{39}}{3} \approx 1.9183$. To find the maximum value of $V(h)$ we evaluate it at the end points and critical number:

$$V(0) = 0, \qquad V\left(\frac{12-\sqrt{39}}{3}\right) \approx 120.1644, \qquad V(5) = 0$$

Therefore the maximum value of the volume must occur at $h = \frac{12-\sqrt{39}}{3} \approx 1.9183$ in and this value is ≈ 120.1644 in^3.

Critical-Point Theorem: By the above, $V'(h) = 140 - 96h + 12h^2$ and the only critical number that we must consider is $h = \frac{12-\sqrt{39}}{3}$. It is easy to see that $V'(h) > 0$ for a$h < \frac{12-\sqrt{39}}{3}$ and $V'(h) < 0$ for $h > \frac{12-\sqrt{39}}{3}$ from $[0,5]$. Therefore the maximum value of the volume must occur at $h = \frac{12-\sqrt{39}}{3} \approx 1.9183$ in and this value is $V\left(\frac{12-\sqrt{39}}{3}\right) \approx 120.1644$ in^3.

EXAMPLE: The annual cost (in thousands of dollars) of manufacturing x thousand sets of wireless earbuds is given by

$$C(x) = .001x^3 + 3x + 100$$

and no more than 60,000 earbuds can be produced in a year.

(a) Find the average cost function.

Solution: The average cost function $\overline{C}$ is given by

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{.001x^3 + 3x + 100}{x} = \frac{.001x^3}{x} + \frac{3x}{x} + \frac{100}{x} = .001x^2 + 3 + 100x^{-1}$$

(b) How many sets of earbuds should be made in order to minimize the average cost per set? What is the minimum average cost?

Solution: Since $x > 0$, we can't use the Closed Interval Method. However, the Critical-Point Theorem can be applied. Indeed, the derivative of $\overline{C}(x)$ is

$$\begin{aligned}\overline{C}'(x) = (.001x^2 + 3 + 100x^{-1})' &= (.001x^2)' + (3)' + (100x^{-1})' \\ &= .001(x^2)' + (3)' + 100(x^{-1})' \\ &= .001(2x) + 0 + 100(-1)x^{-1-1} \\ &= .002x - 100x^{-2}\end{aligned}$$

The critical numbers occur at the numbers in the domain of the average cost function where the value of the derivative $\overline{C}'(x)$ is zero or does not exist. Since $\overline{C}'(x)$ always exists for $0 < x \leq 60$, the only critical numbers occur where $\overline{C}'(x)$ is zero. We have

$$\begin{aligned}.002x - 100x^{-2} &= 0 \\ .002x &= 100x^{-2} \\ .002x \cdot x^2 &= 100x^{-2} \cdot x^2 \\ .002x^3 &= 100 \\ x^3 &= \frac{100}{.002} = 50,000 \\ x &= \sqrt[3]{50,000} \approx 36.84\end{aligned}$$

It is easy to see that $\overline{C}'(x) < 0$ for $0 < x < \sqrt[3]{50,000}$ and $\overline{C}'(x) > 0$ for $x > \sqrt[3]{50,000}$. Therefore by the Critical-Point Theorem 36,840 sets of earbuds should be made in order to minimize the average cost per set. The minimum average cost is $\overline{C}(36.84) = 7.072$. In other words, the minimum average cost is \$7072 per 1000 earbuds or \$7.07 per set of earbuds.