Section 12.2 The Second Derivative

Higher derivatives

If f is a differentiable function, then f' is also a function. So, f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f.

OTHER NOTATIONS:

$$\boxed{f''(x) = y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}}$$

The **third derivative** f''' is the derivative of the second derivative $f''' = (f'')'$.

OTHER NOTATIONS:

$$\boxed{f'''(x) = y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}}$$

Similarly, the **fourth derivative** f'''' is the derivative of the third derivative $f'''' = (f''')'$. And so on.

REMARK: The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general,

$$\boxed{f^{(n)}(x) = y^{(n)} = \frac{d^ny}{dx^n}}$$

EXAMPLES:

(a) Let $f(x) = 2x^3 - 5x$, then

$$f'(x) = (2x^3 - 5x)' = (2x^3)' - (5x)' = 2(x^3)' - 5(x)' = 2(3x^2) - 5(1) = 6x^2 - 5$$

$$f''(x) = (f'(x))' = (6x^2 - 5)' = (6x^2)' - (5)' = 6(x^2)' - (5)' = 6(2x) - 0 = 12x$$

$$f'''(x) = (f''(x))' = (12x)' = 12(x)' = 12(1) = 12$$

$$f^{(4)}(x) = (f'''(x))' = (12)' = 0$$

(b) Let $f(x) = \dfrac{x+2}{5x-1}$, then $f'(x)$ and $f''(x)$ are

(b) Let $f(x) = \dfrac{x+2}{5x-1}$, then

$$\begin{aligned} f'(x) = \left(\frac{x+2}{5x-1}\right)' &= \frac{(x+2)'(5x-1)-(x+2)(5x-1)'}{(5x-1)^2} \\ &= \frac{(x'+2')(5x-1)-(x+2)((5x)'-1')}{(5x-1)^2} \\ &= \frac{(x'+2')(5x-1)-(x+2)(5(x)'-1')}{(5x-1)^2} \\ &= \frac{(1+0)(5x-1)-(x+2)(5(1)-0)}{(5x-1)^2} \\ &= \frac{(1)(5x-1)-(x+2)(5)}{(5x-1)^2} \\ &= \frac{5x-1-5x-10}{(5x-1)^2} \\ &= \frac{-11}{(5x-1)^2} \end{aligned}$$

and

$$\begin{aligned} f''(x) = (f'(x))' = \left(\frac{-11}{(5x-1)^2}\right)' &= \left(-11(5x-1)^{-2}\right)' \\ &= -11\left((5x-1)^{-2}\right)' \\ &= -11(-2)(5x-1)^{-2-1}\cdot(5x-1)' \\ &= -11(-2)(5x-1)^{-3}\cdot((5x)'-(1)') \\ &= -11(-2)(5x-1)^{-3}\cdot(5(x)'-(1)') \\ &= -11(-2)(5x-1)^{-3}\cdot(5(1)-0) \\ &= -11(-2)(5x-1)^{-3}\cdot(5) \\ &= 110(5x-1)^{-3} \\ &= \frac{110}{(5x-1)^3} \end{aligned}$$

(c) Let $f(x) = e^x + x\ln x$, then

$$\begin{aligned} f'(x) = (e^x + x\ln x)' = (e^x)' + (x\ln x)' &= (e^x)' + (x)'\ln x + x(\ln x)' \\ &= e^x + (1)\ln x + x\left(\frac{1}{x}\right) = e^x + \ln x + 1 \end{aligned}$$

and

$$f''(x) = (f'(x))' = (e^x + \ln x + 1)' = (e^x)' + (\ln x)' + (1)' = e^x + \frac{1}{x} + 0 = e^x + \frac{1}{x}$$

Applications

EXAMPLE: Suppose that, over a 10-month period, the price of stock A is given by $f(x) = x^{1/2} + 5$ and the price of stock B is given by $g(x) = .1x^{3/2} + 5$.

(a) When are stocks A and B increasing in price?

Solution: The first derivatives of the price functions are, respectively,

$$f'(x) = (x^{1/2} + 5)' = (x^{1/2})' + (5)' = \frac{1}{2}x^{1/2-1} + 0 = \frac{1}{2}x^{-1/2}$$

and

$$g'(x) = (.1x^{3/2} + 5)' = (.1x^{3/2})' + (5)' = .1(x^{3/2})' + (5)' = .1\left(\frac{3}{2}\right)x^{3/2-1} + 0 = \frac{.3}{2}x^{1/2}$$

Both derivatives are always positive (because $x^{-1/2}$ and $x^{1/2}$ are positive for $x > 0$). So both price functions are increasing for all $x > 0$.

(b) At what rate are these stock prices increasing in the 10th month? What does this suggest about their future performance?

Solution: The rate at which $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = \frac{.3}{2}x^{1/2}$ are increasing is given by their derivatives (the second derivatives of f and g):

$$f''(x) = (f'(x))' = \left(\frac{1}{2}x^{-1/2}\right)' = \frac{1}{2}\left(x^{-1/2}\right)' = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-1/2-1} = -\frac{1}{4}x^{-3/2} = -\frac{1}{4\sqrt{x^3}}$$

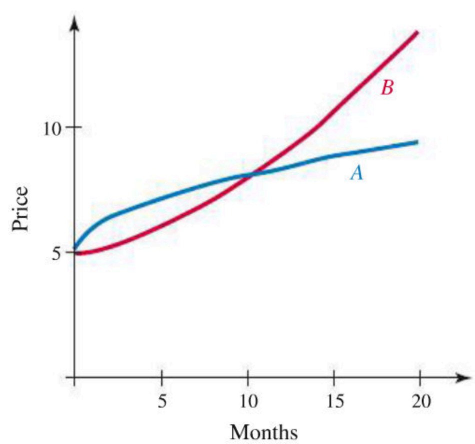
and

$$g''(x) = (g'(x))' = \left(\frac{.3}{2}x^{1/2}\right)' = \frac{.3}{2}\left(x^{1/2}\right)' = \frac{.3}{2}\left(\frac{1}{2}\right)x^{1/2-1} = \frac{.3}{4}x^{-1/2} = \frac{.3}{4\sqrt{x}} = \frac{3}{40\sqrt{x}}$$

When $x = 10$,

$$f''(10) = -\frac{1}{4\sqrt{10^3}} \approx -.0079 \quad \text{and} \quad g''(10) = \frac{3}{40\sqrt{10}} \approx .0237$$

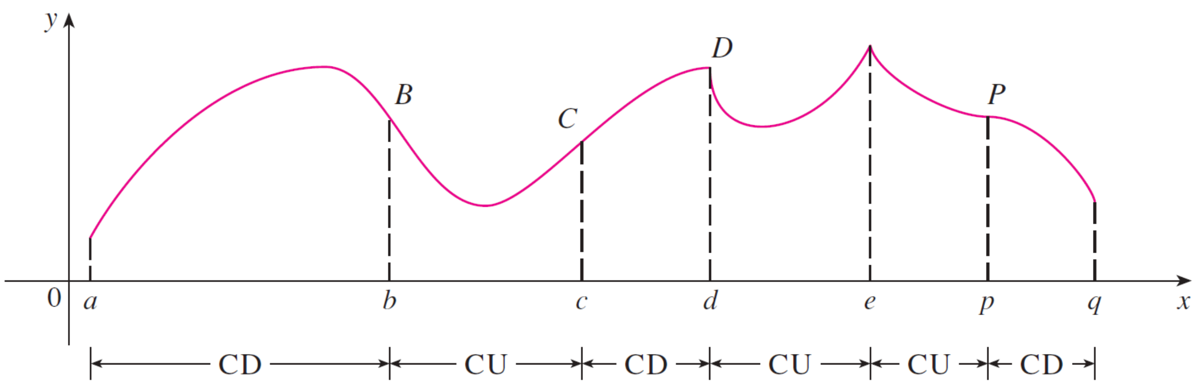
The rate of increase for stock A is negative, meaning that its price is increasing at a decreasing rate. The rate of increase for stock B is positive, meaning that its price is increasing at an increasing rate. In other words, the price of stock A is increasing more and more slowly, while the price of stock B is increasing faster and faster. This result suggests that stock B is probably a better investment for the future.



The preceding discussion assumes that present trends continue (which is not guaranteed in the stock market). If they do continue for another 10 months, then the Figure on the right shows what will happen to the stocks during this period.

DEFINITION: If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on an interval I, then it is called **concave downward** on I.

DEFINITION: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



EXAMPLE: In the picture above, B, C, D, and P are the points of inflection.

CONCAVITY TEST:

(a) If $f''(x) > 0$ on an (open) interval I, then the graph of f is concave upward on I.

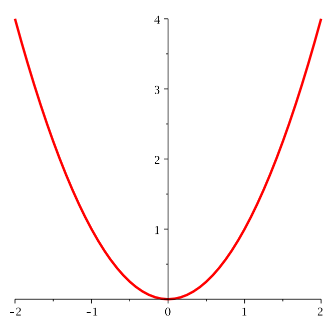
(b) If $f''(x) < 0$ on an (open) interval I, then the graph of f is concave downward on I.

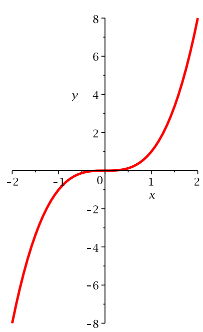
EXAMPLES:

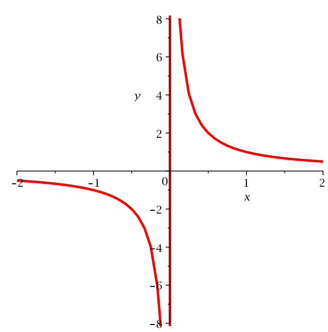
1. The function $f(x) = x^2$ is concave upward on $(-\infty, \infty)$, because $f''(x) = 2$ is positive on $(-\infty, \infty)$. There are no inflection points.

2. The function $f(x) = x^3$ is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$ because $f''(x) = 6x$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. The point $x = 0$ is the inflection point.

3. The function $f(x) = \dfrac{1}{x}$ is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$ because $f''(x) = \dfrac{2}{x^3}$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. There are no inflection points.







EXAMPLE: Find intervals of concavity and the inflection points of $f(x) = x^4 - 4x^3 + 4x^2$.

Solution: We have

$$\begin{aligned} f'(x) = (x^4 - 4x^3 + 4x^2)' &= (x^4)' - (4x^3)' + (4x^2)' \\ &= (x^4)' - 4(x^3)' + 4(x^2)' \\ &= (4x^3) - 4(3x^2) + 4(2x) \\ &= 4x^3 - 12x^2 + 8x \end{aligned}$$

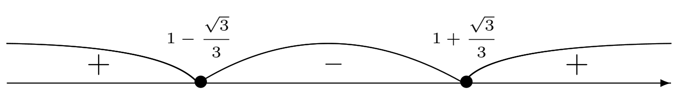
and

$$\begin{aligned} f''(x) = (f'(x))' = (4x^3 - 12x^2 + 8x)' &= (4x^3)' - (12x^2)' + (8x)' \\ &= 4(x^3)' - 12(x^2)' + 8(x)' \\ &= 4(3x^2) - 12(2x) + 8(1) \\ &= 12x^2 - 24x + 8 \\ &= 4(3x^2 - 6x + 2) \end{aligned}$$

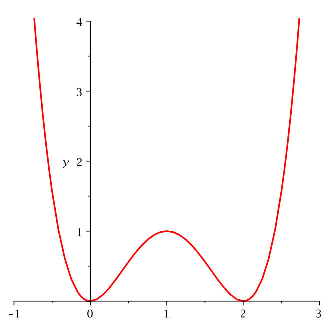
Solving the quadratic equation $3x^2 - 6x + 2 = 0$, we obtain that $f''(x) = 0$ at

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} \\ &= \frac{6 \pm \sqrt{12}}{6} = \left\{ \frac{6}{6} \pm \frac{\sqrt{12}}{6} = 1 \pm \frac{\sqrt{4 \cdot 3}}{6} = 1 \pm \frac{\sqrt{4}\sqrt{3}}{6} = 1 \pm \frac{2\sqrt{3}}{2 \cdot 3} \right\} = 1 \pm \frac{\sqrt{3}}{3} \end{aligned}$$

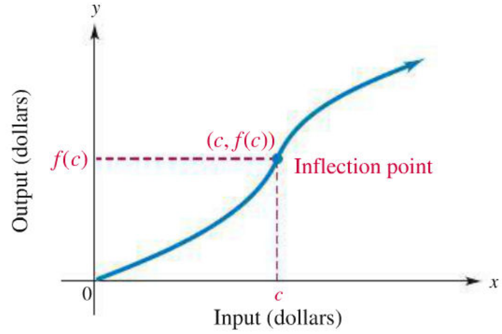
We have



Therefore f is concave upward on $(-\infty, 1 - \sqrt{3}/3)$ and $(1 + \sqrt{3}/3, \infty)$; it is concave downward on $(1 - \sqrt{3}/3, 1 + \sqrt{3}/3)$. The points $1 \pm \sqrt{3}/3$ are the inflection points since the curve changes from concave upward to concave downward and vice-versa there.



The **law of diminishing returns** in economics is related to the idea of concavity. The graph of the function f in the Figure below shows the output y from a given input x. For instance, the input might be advertising costs and the output the corresponding revenue from sales.



The graph in the Figure above shows an inflection point at $(c, f(c))$. For $x < c$, the graph is concave upward, so the rate of change of the slope is increasing. This indicates that the output y is increasing at a faster rate with each additional dollar spent. When $x > c$, however, the graph is concave downward, the rate of change of the slope is decreasing, and the increase in y is smaller with each additional dollar spent. Thus, further input beyond c dollars produces diminishing returns. The point of inflection at $(c, f(c))$ is called the **point of diminishing returns**. Any investment beyond the value c is not considered a good use of capital.

EXAMPLE: The revenue $R(x)$ generated from airline passengers is related to the number of available seat miles x by

$$R(x) = -.928x^3 + 31.492x^2 - 326.80x + 1143.88 \quad (8 \le x \le 12)$$

where x is measured in hundreds of billions of miles and $R(x)$ is in billions of dollars. Is there a point of diminishing returns for this function? If so, what is it? (Data from: Air Transport Association of America.)

Solution: Since a point of diminishing returns occurs at an inflection point, we must find $R''(x)$. We have

$$\begin{aligned} R'(x) &= (-.928x^3 + 31.492x^2 - 326.80x + 1143.88)' \\ &= (-.928x^3)' + (31.492x^2)' - (326.80x)' + (1143.88)' \\ &= -.928(x^3)' + 31.492(x^2)' - 326.80(x)' + (1143.88)' \\ &= -.928(3x^2) + 31.492(2x) - 326.80(1) + 0 \\ &= -2.784x^2 + 62.984x - 326.80 \end{aligned}$$

$$\begin{aligned} R''(x) = (R'(x))' &= (-2.784x^2 + 62.984x - 326.80)' \\ &= (-2.784x^2)' + (62.984x)' - (326.80)' \\ &= -2.784(x^2)' + 62.984(x)' - (326.80)' \\ &= -2.784(2x) + 62.984(1) - 0 \\ &= -5.568x + 62.984 \end{aligned}$$

which exist for every x. When we set $R''(x) = 0$ and solve for x, we obtain

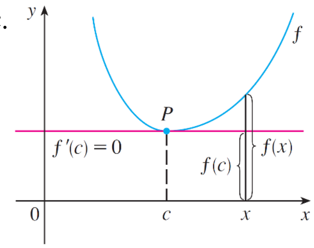
$$-5.568x + 62.984 = 0 \quad \Longrightarrow \quad -5.568x = -62.984 \quad \Longrightarrow \quad x = \frac{-62.984}{-5.568} \approx 11.31$$

Test a number in the interval $(8, 11.31)$ to see that $R''(x)$ is positive there. Then test a number in the interval $(11.31, 12)$ to see that $R''(x)$ is negative in that interval. Since the sign of $R''(x)$ changes from positive to negative at $x = 11.31$, the graph changes from concave upward to concave downward at that point, and there is a point of diminishing returns at the inflection point $(11.31, 133.54)$. Increasing capacity beyond 1.131 trillion seat miles would not pay off.

THE SECOND DERIVATIVE TEST: Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



EXAMPLES:

1. Let $f(x)=x^2$. Since $f'(0)=0$ and $f''(0)=2>0$, it follows that f has a local minimum at 0.

2. Let $f(x)=x$. Since $f'(x)=1$, the test is inconclusive. Note, that $f(x)$ does not have local extreme values by the First Derivative Test.

3. Let $f(x)=x^4$. Since $f'(0)=0$ and $f''(0)=0$, the test is inconclusive. Note, that $f(x)$ has the local minimum at $x=0$ by the First Derivative Test.

EXAMPLE: Discuss the curve $f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

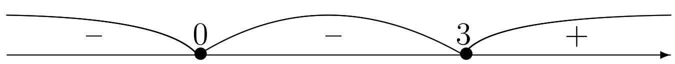
Solution: We have

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3), \quad f''(x) = 12x^2 - 24x = 12x(x-2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. To use the Second Derivative Test we evaluate f'' at these critical numbers:

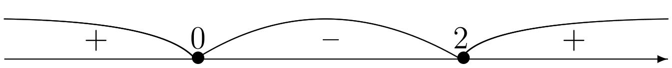
$$f''(0) = 0, \quad f''(3) = 36 > 0$$

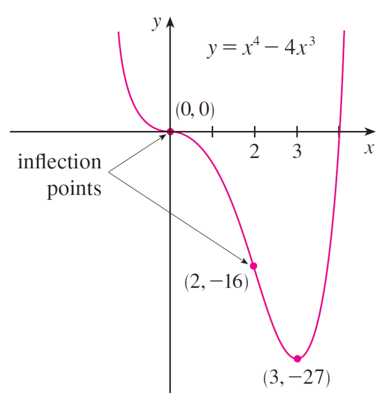
Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0:



Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward





The point $(0, 0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also, $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

EXAMPLE: The number of bicycle trips (in thousands) made in month x by Capital Bikeshare members in Washington, DC can be approximated by

$$g(x) = .27x^3 - 7.5x^2 + 63x - 42 \quad (1 \le x \le 17)$$

where $x = 1$ corresponds to the month of January, 2011. Find all local extrema of this function, and interpret your answers. (Data from: www.capitalbikeshare.com.)

Solution First find the critical numbers. The first derivative is

$$\begin{aligned} g'(x) &= (.27x^3 - 7.5x^2 + 63x - 42)' \\ &= (.27x^3)' - (7.5x^2)' + (63x)' - (42)' \\ &= .27(x^3)' - 7.5(x^2)' + 63(x)' - (42)' \\ &= .27(3x^2) - 7.5(2x) + 63(1) - 0 \\ &= .81x^2 - 15x + 63 \end{aligned}$$

Since $g'(x)$ always exists, critical numbers can occur only where $g'(x) = 0$. We find the solutions to $.81x^2 - 15x + 63 = 0$ using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-15) \pm \sqrt{(-15)^2 - 4(.81)(63)}}{2(.81)} = \frac{15 \pm \sqrt{20.88}}{1.62} \approx 6.4, 12.1$$

The critical numbers are approximately 6.4 and 12.1. Now use the second-derivative test on each critical number. The second derivative is

$$\begin{aligned} g''(x) &= (.81x^2 - 15x + 63)' \\ &= (.81x^2)' - (15x)' + (63)' \\ &= .81(x^2)' - 15(x)' + (63)' \\ &= .81(2x) - 15(1) + 0 \\ &= 1.62x - 15 \end{aligned}$$

For the critical number 6.4, we have

$$g''(6.4) = 1.62(6.4) - 15 = -4.632 < 0$$

which means that there is a local maximum at $x = 6.4$. The local maximum value is

$$g(6.4) = .27(6.4)^3 - 7.5(6.4)^2 + 63(6.4) - 42 \approx 124.8$$

For the critical number 12.1, we have

$$g''(12.1) = 1.62(12.1) - 15 \approx 4.6 > 0$$

hence there is a local minimum at $x = 12.1$. The local minimum value is

$$g(12.1) \approx 100.5$$

In practical terms, this means that Bikeshare use was at a high of 124,800 trips near the middle of June ($x = 6.4$) and at a low of 100,500 trips near the beginning of December ($x = 12.1$).

Appendix I

EXAMPLE: Let $f(x)=\sqrt{1+x^3}$. Find f', f'', and f'''.

Solution: Since $f(x)=(1+x^3)^{1/2}$, we have

$$f'(x)=\frac{1}{2}(1+x^3)^{1/2-1}\cdot(1+x^3)'=\frac{1}{2}(1+x^3)^{-1/2}\cdot 3x^2=\boxed{\frac{3x^2}{2\sqrt{1+x^3}}}$$

$$f''(x)=\left(\frac{3x^2}{2(1+x^3)^{1/2}}\right)'=\frac{3}{2}\left(\frac{x^2}{(1+x^3)^{1/2}}\right)'$$

$$=\frac{3}{2}\cdot\frac{(x^2)'(1+x^3)^{1/2}-x^2[(1+x^3)^{1/2}]'}{[(1+x^3)^{1/2}]^2}$$

$$=\frac{3}{2}\cdot\frac{2x(1+x^3)^{1/2}-x^2\frac{1}{2}(1+x^3)^{1/2-1}\cdot(1+x^3)'}{1+x^3}$$

$$=\frac{3}{2}\cdot\frac{2x(1+x^3)^{1/2}-x^2\frac{1}{2}(1+x^3)^{-1/2}\cdot 3x^2}{1+x^3}$$

$$=\frac{3}{2}\cdot\frac{2x(1+x^3)^{1/2}-\frac{3}{2}x^4(1+x^3)^{-1/2}}{1+x^3}$$

$$=\frac{3}{2}\cdot\frac{\left(2x(1+x^3)^{1/2}-\frac{3}{2}x^4(1+x^3)^{-1/2}\right)\cdot 2(1+x^3)^{1/2}}{(1+x^3)\cdot 2(1+x^3)^{1/2}}$$

$$=\frac{3}{2}\cdot\frac{2x(1+x^3)^{1/2}\cdot 2(1+x^3)^{1/2}-\frac{3}{2}x^4(1+x^3)^{-1/2}\cdot 2(1+x^3)^{1/2}}{2(1+x^3)^{3/2}}$$

$$=\frac{3}{2}\cdot\frac{4x(1+x^3)-3x^4}{2(1+x^3)^{3/2}}$$

$$=\frac{3}{2}\cdot\frac{4x+4x^4-3x^4}{2(1+x^3)^{3/2}}$$

$$=\frac{3}{2}\cdot\frac{4x+x^4}{2(1+x^3)^{3/2}}$$

$$=\frac{3}{2}\cdot\frac{x(4+x^3)}{2(1+x^3)^{3/2}}$$

$$=\boxed{\frac{3x(4+x^3)}{4(1+x^3)^{3/2}}}$$

$$
\begin{aligned}
f'''(x) &= \left(\frac{3x(4+x^3)}{4(1+x^3)^{3/2}} \right)' \\
&= \frac{3}{4} \left(\frac{x(4+x^3)}{(1+x^3)^{3/2}} \right)' \\
&= \frac{3}{4} \cdot \frac{[x(4+x^3)]'(1+x^3)^{3/2} - x(4+x^3)[(1+x^3)^{3/2}]'}{[(1+x^3)^{3/2}]^2} \\
&= \frac{3}{4} \cdot \frac{[x'(4+x^3) + x(4+x^3)'](1+x^3)^{3/2} - x(4+x^3)\frac{3}{2}(1+x^3)^{3/2-1} \cdot (1+x^3)'}{(1+x^3)^3} \\
&= \frac{3}{4} \cdot \frac{[1 \cdot (4+x^3) + x \cdot 3x^2](1+x^3)^{3/2} - x(4+x^3)\frac{3}{2}(1+x^3)^{1/2} \cdot 3x^2}{(1+x^3)^3} \\
&= \frac{3}{4} \cdot \frac{(4+x^3+3x^3)(1+x^3)^{3/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}}{(1+x^3)^3} \\
&= \frac{3}{4} \cdot \frac{(4+4x^3)(1+x^3)^{3/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}}{(1+x^3)^3} \\
&= \frac{3}{4} \cdot \frac{4(1+x^3)(1+x^3)^{3/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}}{(1+x^3)^3} \\
&= \frac{3}{4} \cdot \frac{4(1+x^3)^{5/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}}{(1+x^3)^3} \\
&= \frac{3}{4} \cdot \frac{\left(4(1+x^3)^{5/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2}\right) \cdot 2(1+x^3)^{-1/2}}{(1+x^3)^3 \cdot 2(1+x^3)^{-1/2}} \\
&= \frac{3}{4} \cdot \frac{4(1+x^3)^{5/2} \cdot 2(1+x^3)^{-1/2} - \frac{9}{2}x^3(4+x^3)(1+x^3)^{1/2} \cdot 2(1+x^3)^{-1/2}}{2(1+x^3)^{5/2}} \\
&= \frac{3}{4} \cdot \frac{8(1+x^3)^2 - 9x^3(4+x^3)}{2(1+x^3)^{5/2}} = \frac{3}{4} \cdot \frac{8(1+2x^3+x^6) - 36x^3 - 9x^6}{2(1+x^3)^{5/2}} \\
&= \frac{3}{4} \cdot \frac{8+16x^3+8x^6-36x^3-9x^6}{2(1+x^3)^{5/2}} = \frac{3}{4} \cdot \frac{8-20x^3-x^6}{2(1+x^3)^{5/2}} \\
&= \frac{3(8-20x^3-x^6)}{8(1+x^3)^{5/2}} = \boxed{-\frac{3(x^6+20x^3-8)}{8(1+x^3)^{5/2}}}
\end{aligned}
$$

Appendix II

EXAMPLE: Let $f(x)=3x^2$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:
(a) We have

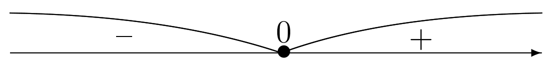
$$f'(x)=(3x^2)'=3(x^2)'=3\cdot 2x=6x$$

Since

$$6x=0 \quad\Longleftrightarrow\quad x=0$$

it follows that the critical number of f is $x=0$.

(b) We have



Therefore f is increasing on $(0,\infty)$; it is decreasing on $(-\infty,0)$.

(c) Because $f'(x)$ changes from negative to positive at 0, the First Derivative Test tells us that $f(0)=0$ is a local minimum value.

(d) We have

$$f''(x)=(6x)'=6(x)'=6\cdot 1=6$$

Since $f'(0)=0$ and $f''(0)=6>0$, the Second Derivative Test tells us that f has a local minimum at 0.

EXAMPLE: Let $f(x)=7x^2-3x+2$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

EXAMPLE: Let $f(x) = 7x^2 - 3x + 2$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

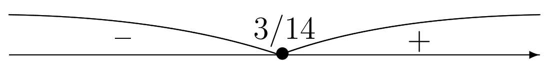
$$f'(x) = (7x^2 - 3x + 2)' = (7x^2)' - (3x)' + 2' = 7(x^2)' - 3(x)' + 2' = 7 \cdot 2x - 3 \cdot 1 + 0 = 14x - 3$$

Since

$$14x - 3 = 0 \iff 14x = 3 \iff x = \frac{3}{14}$$

it follows that the critical number of f is $x = \dfrac{3}{14}$.

(b) We have



Therefore f is increasing on $\left(\dfrac{3}{14}, \infty\right)$; it is decreasing on $\left(-\infty, \dfrac{3}{14}\right)$.

(c) Because $f'(x)$ changes from negative to positive at $\dfrac{3}{14}$, the First Derivative Test tells us that $f\left(\dfrac{3}{14}\right) = \dfrac{47}{28}$ is a local minimum value.

(d) We have

$$f''(x) = (14x - 3)' = (14x)' - 3' = 14(x)' - 3' = 14 \cdot 1 - 0 = 14$$

Since $f'\left(\dfrac{3}{14}\right) = 0$ and $f''\left(\dfrac{3}{14}\right) = 14 > 0$, the Second Derivative Test tells us that f has a local minimum at $\dfrac{3}{14}$.

EXAMPLE: Let $f(x) = 5 - x - 9x^2$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

EXAMPLE: Let $f(x) = 5 - x - 9x^2$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

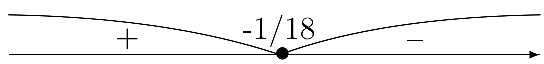
$$f'(x) = (5 - x - 9x^2)' = 5' - x' - (9x^2)' = 5' - x' - 9(x^2)' = 0 - 1 - 9 \cdot 2x = -1 - 18x$$

Since

$$-1 - 18x = 0 \iff -1 = 18x \iff x = -\frac{1}{18}$$

it follows that the critical number of f is $x = -\dfrac{1}{18}$.

(b) We have



Therefore f is increasing on $\left(-\infty, -\dfrac{1}{18}\right)$; it is decreasing on $\left(-\dfrac{1}{18}, \infty\right)$.

(c) Because $f'(x)$ changes from positive to negative at $-\dfrac{1}{18}$, the First Derivative Test tells us that $f\left(-\dfrac{1}{18}\right) = \dfrac{181}{36}$ is a local maximum value.

(d) We have

$$f''(x) = (-1 - 18x)' = (-1)' - (18x)' = (-1)' - 18(x)' = 0 - 18 \cdot 1 = -18$$

Since $f'\left(-\dfrac{1}{18}\right) = 0$ and $f''\left(-\dfrac{1}{18}\right) = -18 < 0$, the Second Derivative Test tells us that f has a local maximum at $-\dfrac{1}{18}$.

EXAMPLE: Let $f(x) = 2x^3 - 27x^2 + 4$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

EXAMPLE: Let $f(x) = 2x^3 - 27x^2 + 4$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

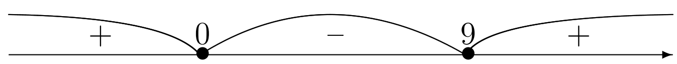
$$f'(x) = (2x^3 - 27x^2 + 4)' = (2x^3)' - (27x^2)' + 4' = 2(x^3)' - 27(x^2)' + 4' = 2 \cdot 3x^2 - 27 \cdot 2x + 0 = 6x^2 - 54x$$

Since

$$6x^2 - 54x = 0 \iff 6x(x - 9) = 0$$

it follows that the critical numbers of f are $x = 0$ and $x = 9$.

(b) We have



Therefore f is increasing on $(-\infty, 0)$ and $(9, \infty)$; it is decreasing on $(0, 9)$.

(c) Because $f'(x)$ changes from positive to negative at 0, the First Derivative Test tells us that $f(0) = 4$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 9, $f(9) = -725$ is a local minimum value.

(d) We have

$$f''(x) = (6x^2 - 54x)' = (6x^2)' - (54x)' = 6(x^2)' - 54(x)' = 6 \cdot 2x - 54 \cdot 1 = 12x - 54$$

Since $f'(0) = 0$ and $f''(0) = -54 < 0$, the Second Derivative Test tells us that f has a local maximum at 0. Similarly, since $f'(9) = 0$ and $f''(9) = 54 > 0$, f has a local minimum at 9.

EXAMPLE: Let $f(x) = 3x - 4x^3$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

EXAMPLE: Let $f(x)=3x-4x^3$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

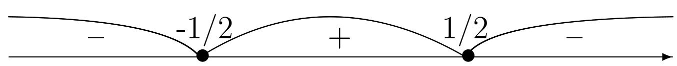
$$f'(x)=(3x-4x^3)'=(3x)'-(4x^3)'=3(x)'-4(x^3)'=3\cdot 1-4\cdot 3x^2=3-12x^2$$

Since

$$3-12x^2=0 \quad\Longleftrightarrow\quad 3=12x^2 \quad\Longleftrightarrow\quad x^2=\frac{3}{12}=\frac{1}{4} \quad\Longleftrightarrow\quad x=\pm\sqrt{\frac{1}{4}}=\pm\frac{1}{2}$$

it follows that the critical numbers of f are $x=\dfrac{1}{2}$ and $x=-\dfrac{1}{2}$.

(b) We have



Therefore f is increasing on $(-1/2,1/2)$; it is decreasing on $(-\infty,-1/2)$ and $(1/2,\infty)$.

(c) Because $f'(x)$ changes from positive to negative at $1/2$, the First Derivative Test tells us that $f\left(\dfrac{1}{2}\right)=1$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at $-1/2$, $f\left(-\dfrac{1}{2}\right)=-1$ is a local minimum value.

(d) We have

$$f''(x)=(3-12x^2)'=3'-(12x^2)'=3'-12(x^2)'=0-12\cdot 2x=-24x$$

Since $f'\left(\dfrac{1}{2}\right)=0$ and $f''\left(\dfrac{1}{2}\right)=-12<0$, the Second Derivative Test tells us that f has a local maximum at $\dfrac{1}{2}$. Similarly, since $f'\left(-\dfrac{1}{2}\right)=0$ and $f''\left(-\dfrac{1}{2}\right)=12>0$, f has a local minimum at $-\dfrac{1}{2}$.

EXAMPLE: Let $f(x)=\dfrac{1}{3}x^3+x^2-15x+1$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

$$f'(x)=\left(\frac{1}{3}x^3+x^2-15x+1\right)'=\left(\frac{1}{3}x^3\right)'+(x^2)'-(15x)'+1'=\frac{1}{3}(x^3)'+(x^2)'-15(x)'+1'$$

$$=\frac{1}{3}\cdot 3x^2+2x-15\cdot 1+0$$

$$=x^2+2x-15$$

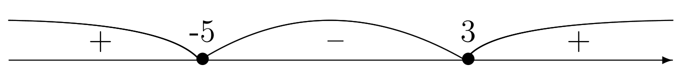
Since

$$x^2+2x-15=0 \iff (x-3)(x+5)=0$$

it follows that the critical numbers of f are $x=3$ and $x=-5$.

(b) We have

Interval	$x-3$	$x+5$	$f'(x)$	f
$x<-5$	$-$	$-$	$+$	increasing on $(-\infty,-5)$
$-5<x<3$	$-$	$+$	$-$	decreasing on $(-5,3)$
$x>3$	$+$	$+$	$+$	increasing on $(3,\infty)$



Therefore f is increasing on $(-\infty,-5)$ and $(3,\infty)$; it is decreasing on $(-5,3)$.

(c) Because $f'(x)$ changes from positive to negative at -5, the First Derivative Test tells us that $f(-5)=\dfrac{178}{3}$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 3, $f(3)=26$ is a local minimum value.

(d) We have

$$f''(x)=(x^2+2x-15)'=(x^2)'+(2x)'-(15)'=(x^2)'+2(x)'-(15)'=2x+2\cdot 1-0=2x+2$$

Since $f'(-5)=0$ and $f''(-5)=-8<0$, the Second Derivative Test tells us that f has a local maximum at -5. Similarly, since $f'(3)=0$ and $f''(3)=8>0$, f has a local minimum at 3.

EXAMPLE: Let $f(x) = 3x^4 - 16x^3 + 24x^2 + 1$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

$$\begin{aligned} f'(x) = \left(3x^4 - 16x^3 + 24x^2 + 1\right)' &= \left(3x^4\right)' - \left(16x^3\right)' + \left(24x^2\right)' + 1' \\ &= 3\left(x^4\right)' - 16\left(x^3\right)' + 24\left(x^2\right)' + 1' \\ &= 3 \cdot 4x^3 - 16 \cdot 3x^2 + 24 \cdot 2x + 0 \\ &= 12x^3 - 48x^2 + 48x \end{aligned}$$

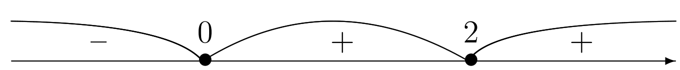
Since

$$12x^3 - 48x^2 + 48x = 0 \quad \Longleftrightarrow \quad 12x(x^2 - 4x + 4) = 0 \quad \Longleftrightarrow \quad 12x(x-2)^2 = 0$$

it follows that the critical numbers of f are $x = 0$ and $x = 2$.

(b) We have

Interval	$12x$	$(x-2)^2$	$f'(x)$	f
$x < 0$	$-$	$+$	$-$	decreasing on $(-\infty, 0)$
$0 < x < 2$	$+$	$+$	$+$	increasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$



Therefore f is increasing on $(0, \infty)$; it is decreasing on $(-\infty, 0)$.

(c) Because $f'(x)$ changes from negative to positive at 0, the First Derivative Test tells us that $f(0) = 1$ is a local minimum value. Since $f'(x)$ does not change its sign at 2, $f(2) = 17$ is not a local extreme value.

(d) We have

$$\begin{aligned} f''(x) = (12x^3 - 48x^2 + 48x)' = (12x^3)' - (48x^2)' + (48x)' &= 12(x^3)' - 48(x^2)' + 48(x)' \\ &= 12 \cdot 3x^2 - 48 \cdot 2x + 48 \cdot 1 \\ &= 36x^2 - 96x + 48 \end{aligned}$$

Since $f'(0) = 0$ and $f''(0) = 48 > 0$, the Second Derivative Test tells us that f has a local maximum at 0. However, since $f'(2) = 0$ and $f''(2) = 0$, the Second Derivative Test test is inconclusive.

EXAMPLE: Let $f(x) = \dfrac{1}{x}$.

(a) Find the critical numbers of f, if any.

(b) Find the intervals on which f is increasing and decreasing.

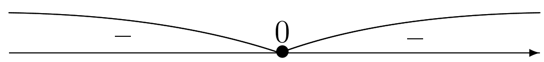
(c) Find the local extreme values of f, if any.

Solution:
(a) We have

$$f'(x) = \left(\frac{1}{x}\right)' = (x^{-1})' = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Note that there are no numbers at which f' is zero. The number at which f' does not exist is $x = 0$, but this number is not from the domain of f. Therefore f does not have critical numbers.

(b) We have



Therefore f is decreasing on $(-\infty, 0)$ and $(0, \infty)$.

(c) Since f does not have critical numbers, it does not have local extreme values.

EXAMPLE: Let $f(x) = \dfrac{3}{x^2}$.

(a) Find the critical numbers of f, if any.

(b) Find the intervals on which f is increasing and decreasing.

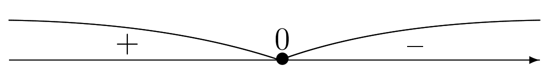
(c) Find the local extreme values of f, if any.

Solution:
(a) We have

$$f'(x) = \left(\frac{3}{x^2}\right)' = (3x^{-2})' = 3(x^{-2})' = 3 \cdot (-2)x^{-2-1} = -6x^{-3} = -\frac{6}{x^3}$$

Note that there are no numbers at which f' is zero. The number at which f' does not exist is $x = 0$, but this number is not from the domain of f. Therefore f does not have critical numbers.

(b) We have



Therefore f is increasing on $(-\infty, 0)$; it is decreasing on $(0, \infty)$.

(c) Since f does not have critical numbers, it does not have local extreme values.

EXAMPLE: Let $f(x) = x + \dfrac{4}{x}$.

(a) Find the critical numbers of f, if any.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f, if any.

Solution:

(a) We have

$$f'(x) = \left(x + \frac{4}{x}\right)' = (x + 4x^{-1})' = x' + (4x^{-1})' = x' + 4(x^{-1})'$$

$$= 1 + 4 \cdot (-1)x^{-1-1} = 1 - 4x^{-2} = 1 - \frac{4}{x^2}$$

Since

$$1 - \frac{4}{x^2} = 0 \iff 1 = \frac{4}{x^2} \iff x^2 = 4 \iff x = \pm 2$$

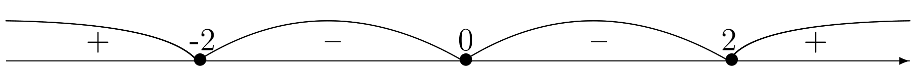
it follows that $f' = 0$ at $x = -2$ and $x = 2$. We also note that the number at which f' does not exist is $x = 0$, but this number is not from the domain of f. Therefore the critical numbers of f are $x = -2$ and $x = 2$ only.

(b) Note that

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2}{x^2} - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{x^2 - 2^2}{x^2} = \frac{(x-2)(x+2)}{x^2}$$

hence

Interval	x^2	$x-2$	$x+2$	$f'(x)$	f
$x < -2$	$+$	$-$	$-$	$+$	increasing on $(-\infty, -2)$
$-2 < x < 0$	$+$	$-$	$+$	$-$	decreasing on $(-2, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$



Therefore f is increasing on $(-\infty, -2)$ and $(2, \infty)$; it is decreasing on $(-2, 0)$ and $(0, 2)$.

(c) Because $f'(x)$ changes from positive to negative at -2, the First Derivative Test tells us that $f(-2) = -4$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 2, $f(2) = 4$ is a local minimum value.

EXAMPLE: Let $f(x) = x \ln x - x$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

Solution:

(a) We have

$$f'(x) = (x \ln x - x)' = (x \ln x)' - x' = x' \ln x + x (\ln x)' - x'$$

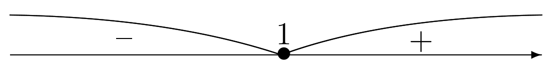
$$= 1 \cdot \ln x + x \cdot \frac{1}{x} - 1 = \ln x + 1 - 1 = \ln x$$

Since

$$\ln x = 0 \quad \iff \quad x = 1$$

it follows that $f' = 0$ at $x = 1$. Therefore the critical number of f is $x = 1$.

(b) We have



Therefore f is decreasing on $(-\infty, 1)$; it is increasing on $(1, \infty)$.

(c) Because $f'(x)$ changes from negative to positive at 1, the First Derivative Test tells us that

$$f(1) = 1 \cdot \ln 1 - 1 = 1 \cdot 0 - 1 = 0 - 1 = -1$$

is a local minimum value.

EXAMPLE: Let $f(x) = 2x \ln x - 3x$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

EXAMPLE: Let $f(x)=2x\ln x-3x$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

Solution:

(a) We have

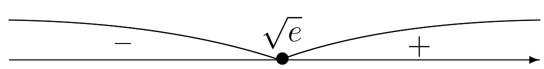
$$\begin{aligned} f'(x)=(2x\ln x-3x)'=(2x\ln x)'-(3x)' &= 2\,(x\ln x)'-3(x)' \\ &= 2(x'\ln x+x\,(\ln x)')-3(x)' \\ &= 2\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)-3\cdot 1 \\ &= 2\,(\ln x+1)-3=2\ln x+2-3=2\ln x-1 \end{aligned}$$

Since

$$2\ln x-1=0 \quad\Longleftrightarrow\quad 2\ln x=1 \quad\Longleftrightarrow\quad \ln x=\frac{1}{2} \quad\Longleftrightarrow\quad x=e^{1/2}=\sqrt{e}$$

it follows that $f'=0$ at $x=\sqrt{e}$. Therefore the critical number of f is $x=\sqrt{e}$.

(b) We have



Therefore f is decreasing on $(-\infty,\sqrt{e})$; it is increasing on $(\sqrt{e},\infty)$.

(c) Because $f'(x)$ changes from negative to positive at $\sqrt{e}$, the First Derivative Test tells us that

$$\begin{aligned} f\left(\sqrt{e}\right)=2e^{1/2}\ln e^{1/2}-3e^{1/2} &= 2e^{1/2}\cdot\frac{1}{2}\ln e-3e^{1/2} \\ &= 2e^{1/2}\cdot\frac{1}{2}\cdot 1-3e^{1/2}=e^{1/2}-3e^{1/2}=-2e^{1/2}=-2\sqrt{e} \end{aligned}$$

is a local minimum value.