Section 12.1 Derivatives and Graphs

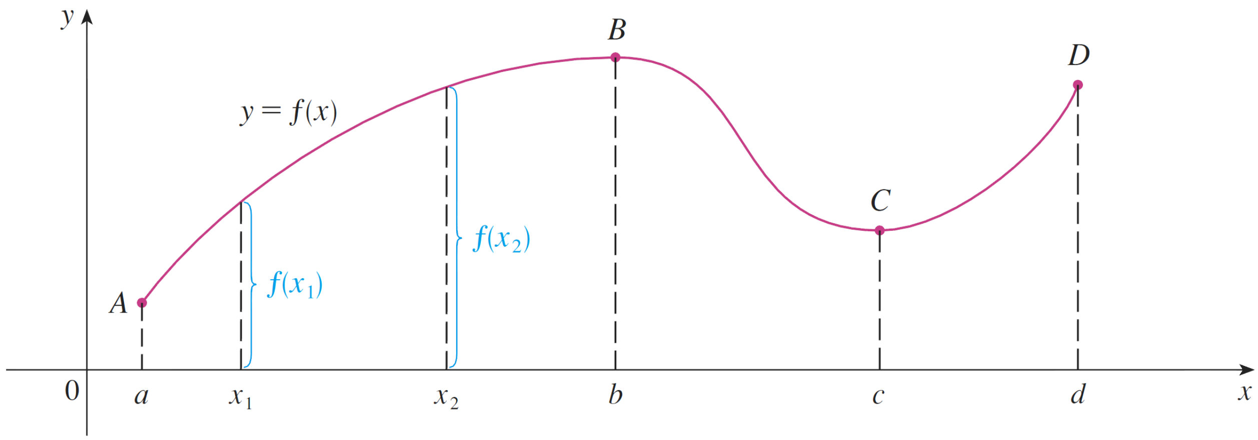
Increasing and Decreasing Functions

DEFINITION: A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on an I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

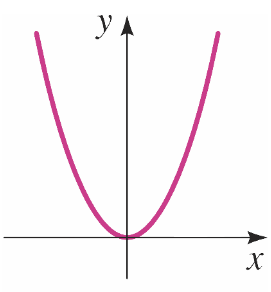


EXAMPLES:

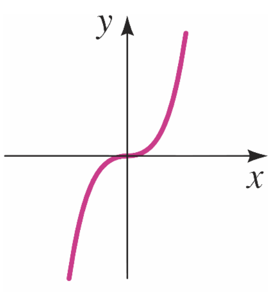
1. The function $f(x) = x^2$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

2. The function $f(x) = x^3$ is increasing everywhere, that is on $(-\infty, \infty)$.

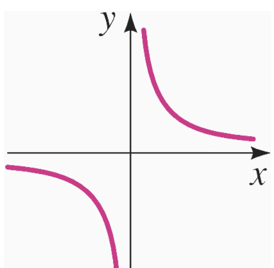
3. The function $f(x) = \dfrac{1}{x}$ is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.



$f(x) = x^2$



$f(x) = x^3$



$f(x) = \dfrac{1}{x}$

INCREASING/DECREASING FUNCTIONS:

(a) If $f'(x) > 0$ on an open interval I, then f is increasing on I.

(b) If $f'(x) < 0$ on an open interval I, then f is decreasing on I.

INCREASING/DECREASING TEST: To find the intervals on which a function f is increasing or decreasing, do the following:

Step 1: Compute the derivative f'.

Step 2: Find all numbers such that f' is zero or does not exist.

Step 3: Solve the inequalities $f'(x) > 0$ and $f'(x) < 0$ (by testing a number in each of the intervals determined by the numbers from Step 2).

The solutions of $f'(x) > 0$ are intervals on which f is increasing, and the solutions of $f'(x) < 0$ are intervals on which f is decreasing.

EXAMPLES:

1. Let $f(x) = 3x + 8$. We have

$$f'(x) = (3x+8)' = (3x)' + 8' = 3(x)' + 8' = 3(1) + 0 = 3$$

therefore f' is positive for any x. It follows that f is increasing on $(-\infty, \infty)$.

2. Let $f(x) = \dfrac{1-2x}{5}$. We have

$$f'(x) = \left(\frac{1-2x}{5}\right)' = \left(\frac{1}{5} - \frac{2x}{5}\right)' = \left(\frac{1}{5} - \frac{2}{5}x\right)'$$

$$= \left(\frac{1}{5}\right)' - \left(\frac{2}{5}x\right)' = \left(\frac{1}{5}\right)' - \frac{2}{5}(x)' = 0 - \frac{2}{5}(1) = -\frac{2}{5}$$

therefore f' is negative for any x. It follows that f is decreasing on $(-\infty, \infty)$.

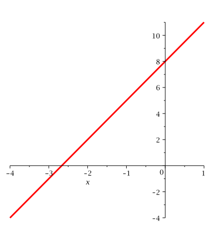
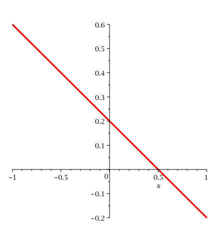
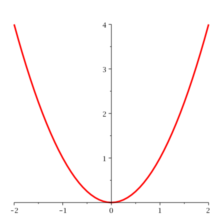
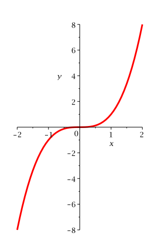
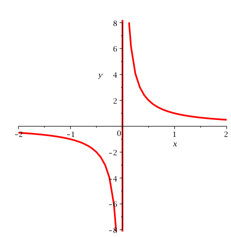
3. The function $f(x) = x^2$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, because $f'(x) = 2x$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$.

4. The function $f(x) = x^3$ is increasing on $(-\infty, \infty)$, because $f'(x) = 3x^2$ is positive (nonnegative) on $(-\infty, \infty)$.

5. The function $f(x) = \dfrac{1}{x}$ is decreasing on $(-\infty, 0)$ and $(0, \infty)$, because

$$f'(x) = (x^{-1})' = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

is negative everywhere except for $x = 0$.

EXAMPLE: Find where the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

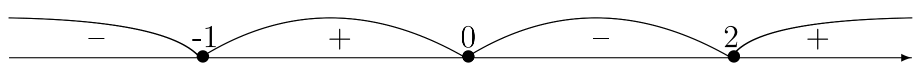
is increasing and where it is decreasing.

Solution: Since

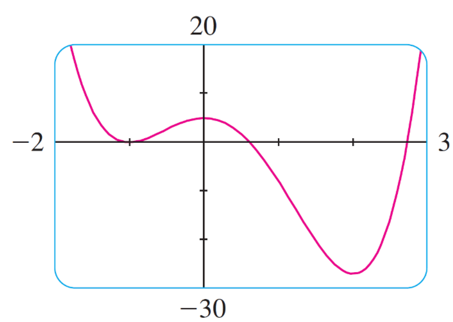
$$\begin{aligned} f'(x) &= (3x^4 - 4x^3 - 12x^2 + 5)' \\ &= (3x^4)' - (4x^3)' - (12x^2)' + (5)' \\ &= 3(x^4)' - 4(x^3)' - 12(x^2)' + (5)' \\ &= 3(4x^3) - 4(3x^2) - 12(2x) + 0 \\ &= 12x^3 - 12x^2 - 24x \\ &= 12x(x^2 - x - 2) \\ &= 12x(x-2)(x+1) \end{aligned}$$

we have

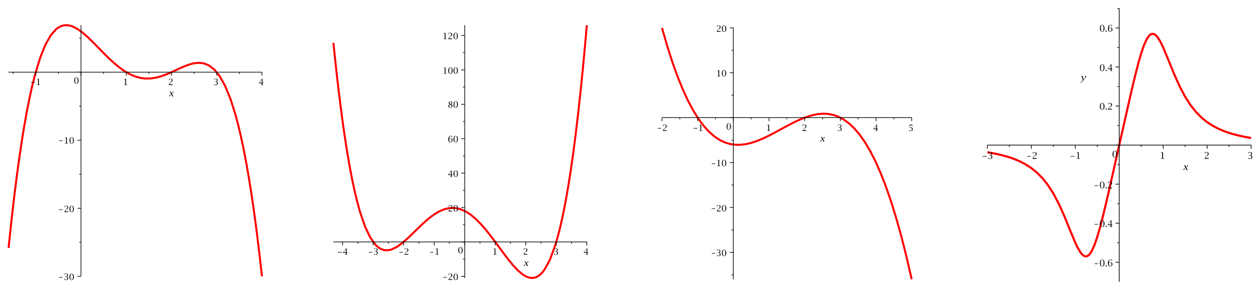
Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$



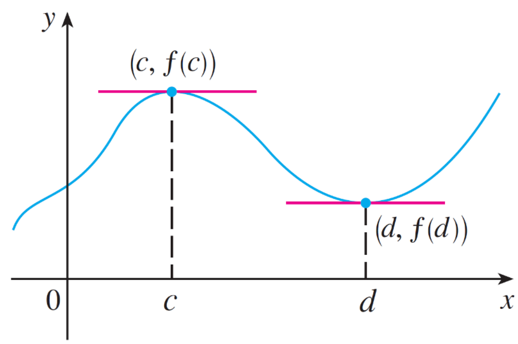
Therefore f is increasing on $(-1, 0)$ and $(2, \infty)$; it is decreasing on $(-\infty, -1)$ and $(0, 2)$.



DEFINITION: A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c. [This means that $f(c) \geq f(x)$ for all x in some open interval containing c.] Similarly, f has a **local minimum** (or **relative minimum**) at c if $f(c) \leq f(x)$ when x is near c.

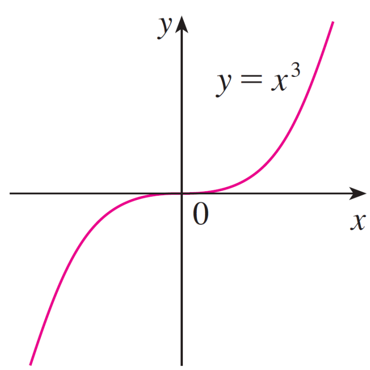


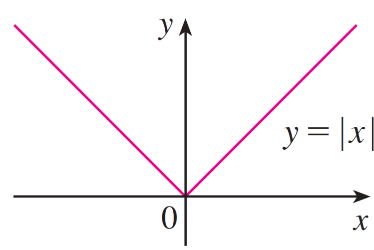
THEOREM (Fermat's Theorem): If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.



REMARK 1: The converse of this theorem is not true. In other words, when $f'(c) = 0$, f does not necessarily have a local maximum or minimum. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ equals 0 at $x = 0$, but $x = 0$ is not a point of a local minimum or maximum.

REMARK 2: Sometimes $f'(c)$ does not exist, but $x = c$ is a point of a local maximum or minimum. For example, if $f(x) = |x|$, then $f'(0)$ does not exist. But $f(x)$ has its local (and absolute) minimum at $x = 0$.





DEFINITION: A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

REMARK: From Fermat's Theorem it follows that if f has a local maximum or minimum at c, then c is a critical number of f.

EXAMPLES:

(a) If $f(x) = 2x^2 + 5x - 1$, then

$$f'(x) = (2x^2+5x-1)' = (2x^2)'+(5x)'-(1)' = 2(x^2)'+5(x)'-(1)' = 2(2x)+5(1)-0 = 4x+5$$

Hence the only critical number of f is $x = -\dfrac{5}{4}$.

(b) If $f(x) = \sqrt[3]{x^2}$, then

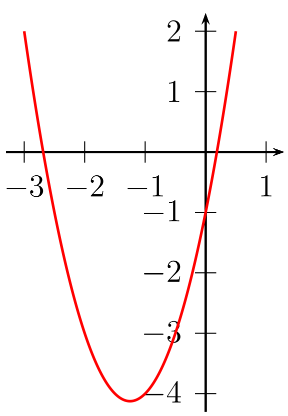
$$f'(x) = (x^{2/3})' = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

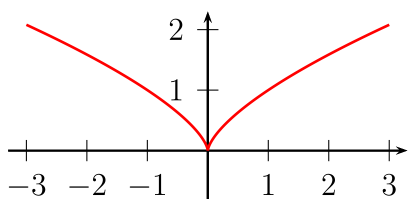
Hence the only critical number of f is $x = 0$.

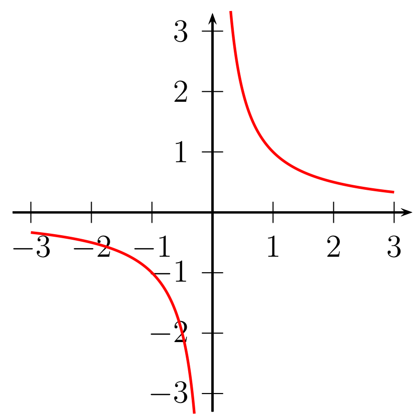
(c) If $f(x) = \dfrac{1}{x}$, then

$$f'(x) = (x^{-1})' = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

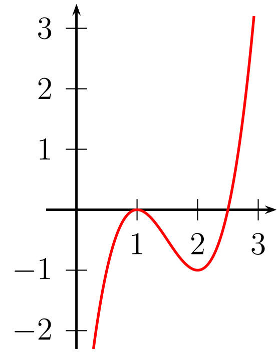
Since $x = 0$ is not in the domain, f has no critical numbers.







EXAMPLES: Find the critical numbers of $f(x) = 2x^3 - 9x^2 + 12x - 5$.



Solution: We have

$$\begin{aligned} f'(x) &= 6x^2 - 18x + 12 \\ &= 6(x^2 - 3x + 2) \\ &= 6(x-1)(x-2) \end{aligned}$$

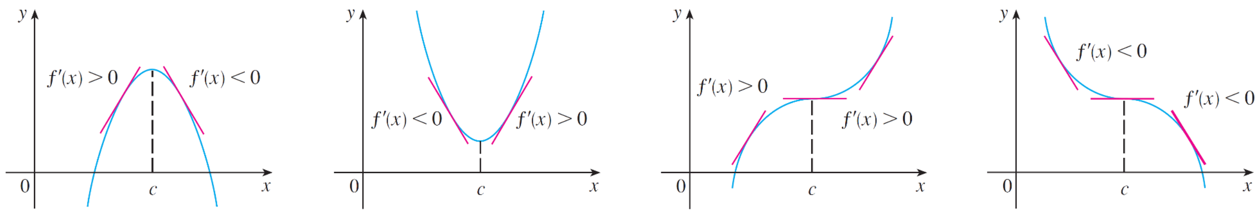
thus $f'(x) = 0$ at $x = 1$ and $x = 2$. Since $f'(x)$ exists everywhere, $x = 1$ and $x = 2$ are the only critical numbers.

THE FIRST DERIVATIVE TEST: Suppose c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

EXAMPLE: Find where

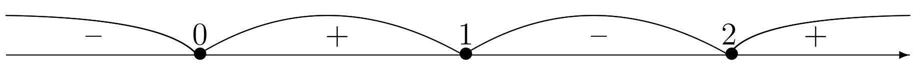
$$f(x) = x^4 - 4x^3 + 4x^2$$

is increasing and where it is decreasing. Find the local maximum and minimum values of f.

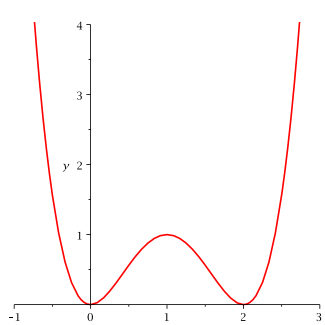
Solution: Since

$$\begin{aligned} f'(x) = (x^4 - 4x^3 + 4x^2)' &= (x^4)' - (4x^3)' + (4x^2)' \\ &= (x^4)' - 4(x^3)' + 4(x^2)' \\ &= (4x^3) - 4(3x^2) + 4(2x) \\ &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x-1)(x-2) \end{aligned}$$

the critical numbers are $x = 0, 1$ and 2. We have



Therefore f is increasing on $(0,1)$ and $(2,\infty)$; it is decreasing on $(-\infty,0)$ and $(1,2)$. Because $f'(x)$ changes from negative to positive at 0 and 2, the First Derivative Test tells us that $f(0) = 0$ and $f(2) = 0$ are local minimum values. Similarly, since $f'(x)$ changes from positive to negative at 1, $f(1) = 1$ is a local maximum value.



EXAMPLE: Let $f(x)=2x+3\sqrt[3]{x^2}$.

(a) Find the critical numbers of f, if any.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f, if any.

Solution:

(a) We have

$$\begin{aligned} f'(x) &= (2x+3x^{2/3})' \\ &= (2x)'+(3x^{2/3})' \\ &= 2(x)'+3(x^{2/3})' \\ &= 2(1)+3\cdot\frac{2}{3}x^{2/3-1} \\ &= 2+2x^{-1/3} \end{aligned}$$

This can be rewritten as $\dfrac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$ in two different ways. Either

$$2+2x^{-1/3}=\frac{2}{1}+\frac{2}{\sqrt[3]{x}}=\frac{2\sqrt[3]{x}}{\sqrt[3]{x}}+\frac{2}{\sqrt[3]{x}}=\frac{2\sqrt[3]{x}+2}{\sqrt[3]{x}}=\frac{2\sqrt[3]{x}+2\cdot 1}{\sqrt[3]{x}}=\frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$$

or

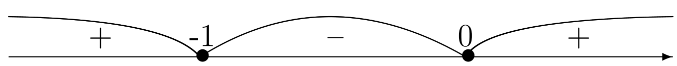
$$2+2x^{-1/3}=2x^{-1/3}\cdot x^{1/3}+2x^{-1/3}\cdot 1=2x^{-1/3}(x^{1/3}+1)=\frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$$

So,

$$f'(x)=\frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$$

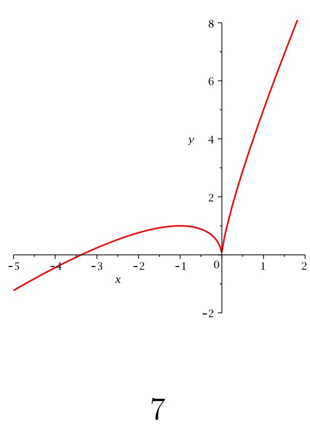
therefore the critical numbers are $x=-1$ (where the top is equal to 0) and 0 (where the bottom is equal to 0).

(b) We have



Therefore f is increasing on $(-\infty,-1)$ and $(0,\infty)$; it is decreasing on $(-1,0)$.

(c) Because $f'(x)$ changes from positive to negative at -1, the First Derivative Test tells us that $f(-1)=1$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 0, $f(0)=0$ is a local minimum value.



EXAMPLE: Let $f(x) = (x-2)e^{-x}$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

Solution:

(a) We have

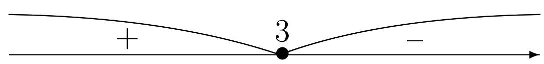
$$\begin{aligned} f'(x) = \Big((x-2)e^{-x}\Big)' &= (x-2)'e^{-x} + (x-2)\left(e^{-x}\right)' \\ &= (x'-2')e^{-x} + (x-2)e^{-x} \cdot (-x)' \\ &= (1-0)e^{-x} + (x-2)e^{-x} \cdot (-1) \\ &= 1 \cdot e^{-x} - (x-2)e^{-x} \\ &= e^{-x}\Big(1-(x-2)\Big) \\ &= e^{-x}(1-x+2) \\ &= e^{-x}(3-x) \end{aligned}$$

Since

$$e^{-x}(3-x) = 0 \quad \Longleftrightarrow \quad 3-x=0 \quad \Longleftrightarrow \quad x=3$$

it follows that $f' = 0$ at $x = 3$. Therefore the critical number of f is $x = 3$.

(b) We have



Therefore f is increasing on $(-\infty, 3)$; it is decreasing on $(3, \infty)$.

(c) Because $f'(x)$ changes from positive to negative at 3, the First Derivative Test tells us that

$$f(3) = (3-2)e^{-3} = 1 \cdot e^{-3} = e^{-3}$$

is a local maximum value.

Applications

EXAMPLE: Sales (in thousands of dollars) for Huttig Building Products, Inc. can be approximated by

$$S(x) = 8x^3 - 180x^2 + 1170x - 1260 \quad (3 \le x \le 11)$$

where $x = 3$ corresponds to the year 2003. Determine when sales were increasing and when sales were decreasing. Also find all local extrema of the sales function. (Data from: www.morningstar.com.)

Solution: To determine the critical numbers, we find the derivative

$$\begin{aligned} S'(x) &= (8x^3 - 180x^2 + 1170x - 1260)' \\ &= (8x^3)' - (180x^2)' + (1170x)' - (1260)' \\ &= 8(x^3)' - 180(x^2)' + 1170(x)' - (1260)' \\ &= 8(3x^2) - 180(2x) + 1170(1) - 0 \\ &= 24x^2 - 360x + 1170 \end{aligned}$$

Since the derivative exists for every x, the only critical numbers occur where the derivative is zero. Solve $S'(x) = 0$ using the quadratic formula:

$$24x^2 - 360x + 1170 = 0$$

hence

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-360) \pm \sqrt{(-360)^2 - 4(24)(1170)}}{2(24)} \approx 4.8, 10.2$$

Since the domain of the sales function is $[3, 11]$, both critical numbers lie within the domain. We choose the interval $[4, 5]$ to test the critical number 4.8, and the interval $[5, 11]$ to test the critical number 10.2:

$$S'(4) = 114 > 0$$

$$S'(5) = -30 < 0$$

$$S'(11) = 114 > 0$$

Therefore, sales are increasing on the intervals $(3, 4.8)$ and $(10.2, 11)$ and decreasing on the interval $(4.8, 10.2)$. By the first-derivative test, there is a local maximum at $x = 4.8$ and a local minimum at $x = 10.2$. In other words, sales reach a local maximum toward the end of 2004 and a local minimum near the beginning of 2010. The local maximum value is $S(4.8) \approx 1093.5$ (sales of approximately \$1,093,500) and the local minimum value is $S(10.2) \approx 436.5$ (sales of approximately \$436,500), as can be seen in the Figure below.