Section 11.9 Continuity and Differentiability

DEFINITION: A function f is **continuous at a number** a if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

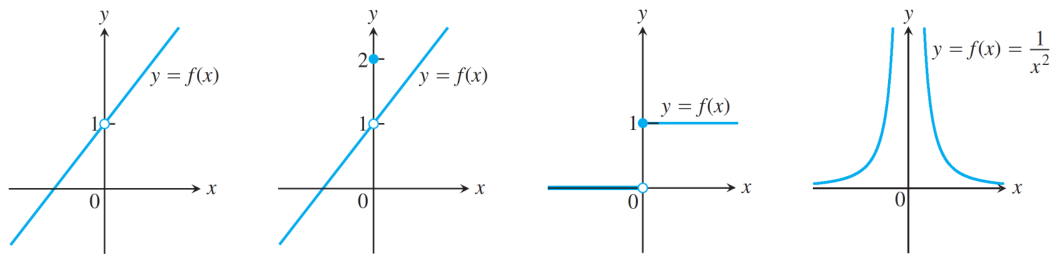
REMARK: It follows from the definition that f is continuous at a if and only if

1. $f(a)$ is defined.

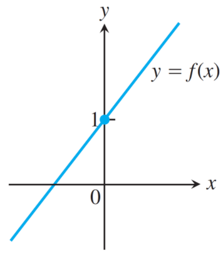
2. $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist.

3. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$.

EXAMPLE: The following functions are not continuous at $x = 0$:



The function



is continuous at $x = 0$.

EXAMPLE:

(a) The function

$$f(x) = \frac{1}{1 - x^2}$$

is discontinuous at $x = \pm 1$, since $f(x)$ is not defined at these points.

(b) The function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \leq 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

is discontinuous at $x = 2$. In fact,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 1) = 2 \cdot 2 - 1 = 3 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 2^2 = 4$$

so

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

therefore $f(x)$ is discontinuous at $x = 2$.

DEFINITION: A function f is **continuous from the right at a number** $\boldsymbol{a}$ if

$$\boxed{\lim_{x\to a^+} f(x) = f(a)}$$

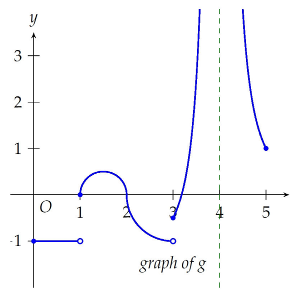
and f is **continuous from the left at** $\boldsymbol{a}$ if

$$\boxed{\lim_{x\to a^-} f(x) = f(a)}$$

DEFINITION: A function f is **continuous on an open interval** (a, b) if it is continuous at every point in the interval. A function f is **continuous on a closed interval** $[a, b]$ if

1. f is continuous at every point in the interval (a, b);
2. f is continuous from the right at $x = a$;
3. f is continuous from the left at $x = b$.

EXAMPLE: The graph of a function g is shown.

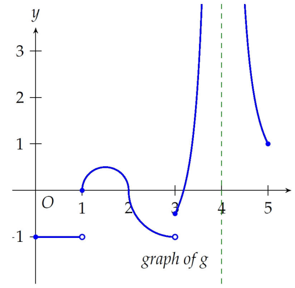


(a) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous?

(b) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous from the right? From the left?

(c) On which intervals $[0, 1)$, $[0, 1]$, $[1, 3]$, $[1, 3)$ is g continuous?

EXAMPLE: The graph of a function g is shown.



(a) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous?

(b) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous from the right? From the left?

(c) On which intervals $[0, 1)$, $[0, 1]$, $[1, 3]$, $[1, 3)$ is g continuous?

Solution:

(a) The function g is continuous at $a = 0, 2, 5$. In fact,

(i) The function g is continuous at $a = 0$, since $\lim\limits_{x\to 0^+} g(x) = g(0) = -1$.

(ii) The function g is not continuous at $a = 1$, since $\lim\limits_{x\to 1^-} g(x) \neq \lim\limits_{x\to 1^+} g(x)$.

(iii) The function g is continuous at $a = 2$, since $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^+} g(x) = g(2) = 0$.

(iv) The function g is not continuous at $a = 3$, since $\lim\limits_{x\to 3^-} g(x) \neq \lim\limits_{x\to 3^+} g(x)$.

(v) The function g is not continuous at $a = 4$, since $g(4)$ does not exist.

(vi) The function g is continuous at $a = 5$, since $\lim\limits_{x\to 5^-} g(x) = g(5) = 1$.

(b) The function g is continuous from the right at $a = 0, 1, 2, 3$. In fact,

(i) The function g is continuous from the right at $a = 0$, since $\lim\limits_{x\to 0^+} g(x) = g(0)$.

(ii) The function g is continuous from the right at $a = 1$, since $\lim\limits_{x\to 1^+} g(x) = g(1)$.

(iii) The function g is continuous from the right at $a = 2$, since $\lim\limits_{x\to 2^+} g(x) = g(2)$.

(iv) The function g is continuous from the right at $a = 3$, since $\lim\limits_{x\to 3^+} g(x) = g(3)$.

(v) The function g is not continuous from the right at $a = 4$, since $g(4)$ does not exist.

(vi) The function g is not continuous from the right at $a = 5$, since $\lim\limits_{x\to 5^+} g(x)$ does not exist.

The function g is continuous from the left at $a = 2, 5$. In fact,

(i) The function g is not continuous from the left at $a = 0$, since $\lim\limits_{x\to 0^-} g(x)$ does not exist.

(ii) The function g is not continuous from the left at $a = 1$, since $\lim\limits_{x\to 1^-} g(x) \neq g(1)$.

(iii) The function g is continuous from the left at $a = 2$, since $\lim\limits_{x\to 2^-} g(x) = g(2)$.

(iv) The function g is not continuous from the left at $a = 3$, since $\lim\limits_{x\to 3^-} g(x) \neq g(3)$.

(v) The function g is not continuous from the left at $a = 4$, since $g(4)$ does not exist.

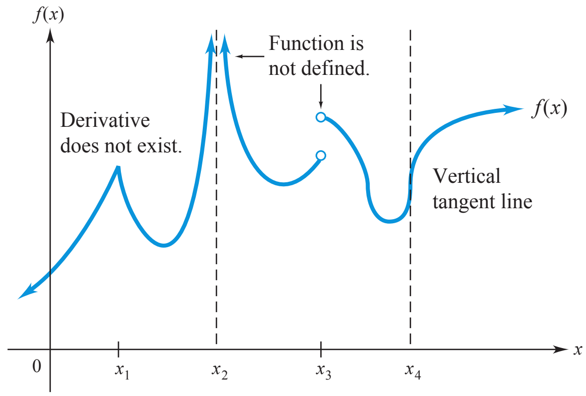
(vi) The function g is continuous from the left at $a = 5$, since $\lim\limits_{x\to 5^-} g(x) = g(5)$.

(c) The function g is continuous on $[0, 1)$ and $[1, 3)$, since it is continuous at every point from the open intervals $(0, 1)$, $(1, 3)$ and continuous from the right at $a = 0$ and $a = 1$. The function g is not continuous on $[0, 1]$ and $[1, 3]$, since it is not continuous from the left at $a = 1$ and $a = 3$.

Continuity and Differentiability

As shown earlier in this chapter, a function fails to have a derivative at a point where the function is not defined, where the graph of the function has a "sharp point," or where the graph has a vertical tangent line. (See the Figure below.)

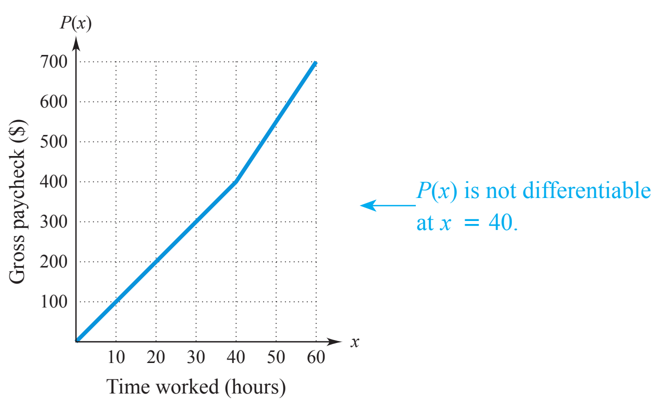
The function graphed in the Figure is continuous on the interval (x_1, x_2) and has a derivative at each point on this interval. On the other hand, the function is also continuous on the interval $(0, x_2)$, but does *not* have a derivative at each point on the interval. (See x_1 on the graph.)



A similar situation holds in the general case.

> If the derivative of a function exists at a point, then the function is continuous at that point. However, a function may be continuous at a point and not have a derivative there.

EXAMPLE: The Fair Labor Standards Act requires overtime pay for covered, nonexempt employees at a rate of not less than one and one-half times an employee's regular rate of pay after 40 hours of work in a workweek. A fast food restaurant pays an entry-level employee \$10 per hour for regular pay and time-and-a-half for overtime pay. Let $P(x)$ represent the employee's gross pay prior to taxes and other withholdings (in dollars) for working x hours in a week. The graph of $y = P(x)$ is shown in the Figure below. Find any points where P is discontinuous. Also find any points where P is not differentiable. (Data from: United States Department of Labor.)

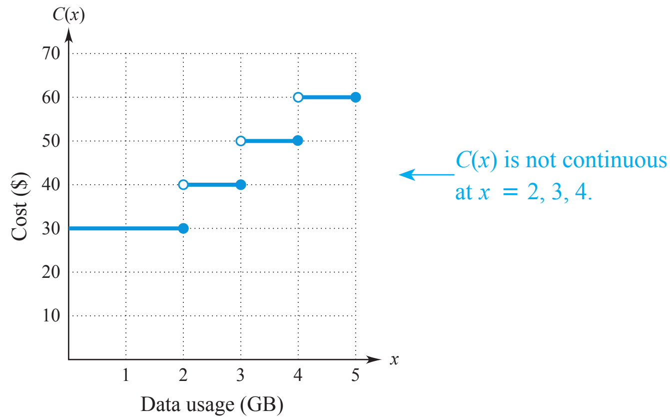


Solution: The graph indicates that P is a continuous function since it can be drawn without lifting pencil from paper. However, P is not differentiable at $x = 40$ hours where the graph has a sharp corner.

EXAMPLE: The monthly cost $C(x)$ (in dollars) of a popular smart phone data plan for the use of x gigabytes of data is \$30 for the first 2GB of data plus \$10 for each additional GB of data (or fraction thereof). (Data from: Verizon Wireless, 2012 prices.)

(a) Sketch the graph of $y = C(x)$ on the interval $0 \leq x \leq 5$.

Solution: For any amount of data usage up to and including 2GB, the charge is \$30. The charge for using more than 2 GB but less than 3 GB of data is \$40. Similar results lead to the graph in the Figure below.



(b) Find any points of discontinuity for C on the interval $(0, 5)$. Also find any points on $(0, 5)$ where C is not differentiable.

Solution: As the graph suggests, C is discontinuous at $x = 2, 3, 4$ gigabytes. However, C is continuous from the left at each of these x-values. Since C is discontinuous at $x = 2, 3, 4$, it is not differentiable at these points.