Section 11.8 Derivatives of Exponential and Logarithmic Functions

BASIC DIFFERENTIATION RULES

$c' = 0, \quad x' = 1$	$(u^n)' = nu^{n-1} \cdot u'$	$[cf(x)]' = cf'(x)$
$(a^u)' = a^u \ln a \cdot u'$	$(e^u)' = e^u \cdot u'$	$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
$(\log_a u)' = \dfrac{1}{u \ln a} \cdot u'$	$(\ln u)' = \dfrac{1}{u} \cdot u'$	$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
		$\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

EXAMPLES:

1. If $f(x) = 2^{3x}$, then

$$f'(x) = (2^{3x})' = 2^{3x} \ln 2 \cdot (3x)' = 2^{3x} \ln 2 \cdot 3$$

2. If $f(x) = xe^{\sqrt{1-x}}$, then

$$\begin{aligned}
f'(x) = (xe^{\sqrt{1-x}})' = x'e^{\sqrt{1-x}} + x(e^{\sqrt{1-x}})' &= 1 \cdot e^{\sqrt{1-x}} + xe^{\sqrt{1-x}} \cdot (\sqrt{1-x})' \\
&= e^{\sqrt{1-x}} + xe^{\sqrt{1-x}} \cdot ((1-x)^{1/2})' \\
&= e^{\sqrt{1-x}} + xe^{\sqrt{1-x}} \cdot \frac{1}{2}(1-x)^{1/2-1} \cdot (1-x)' \\
&= e^{\sqrt{1-x}} + xe^{\sqrt{1-x}} \cdot \frac{1}{2}(1-x)^{-1/2} \cdot (1' - x') \\
&= e^{\sqrt{1-x}} + xe^{\sqrt{1-x}} \cdot \frac{1}{2}(1-x)^{-1/2} \cdot (0 - 1) \\
&= e^{\sqrt{1-x}} + xe^{\sqrt{1-x}} \cdot \frac{1}{2}(1-x)^{-1/2} \cdot (-1) \\
&= e^{\sqrt{1-x}} - \frac{1}{2}xe^{\sqrt{1-x}}(1-x)^{-1/2} \\
&= e^{\sqrt{1-x}} - \frac{xe^{\sqrt{1-x}}}{2\sqrt{1-x}}
\end{aligned}$$

EXAMPLES:

(a) If $f(x) = \log_5(x^2+1)$, then $f'(x) = [\log_5(x^2+1)]' = \dfrac{1}{(x^2+1)\ln 5} \cdot (x^2+1)' = \dfrac{2x}{(x^2+1)\ln 5}$.

(b) If $f(x) = \ln(\ln x)$, then $f'(x) = [\ln(\ln x)]' = \dfrac{1}{\ln x} \cdot (\ln x)' = \dfrac{1}{\ln x} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln x}$.

(c) If $f(x) = \log\left(\frac{x}{1+x^2}\right)$, then

$$f'(x) = \left[\log\left(\frac{x}{1+x^2}\right)\right]' = \frac{1}{\frac{x}{1+x^2}\cdot \ln 10}\cdot\left(\frac{x}{1+x^2}\right)'$$

$$= \left\{\frac{(1+x^2)\cdot 1}{(1+x^2)\cdot\frac{x}{1+x^2}\cdot\ln 10}\cdot\left(\frac{x}{1+x^2}\right)'\right\}$$

$$= \frac{1+x^2}{x\ln 10}\cdot\left(\frac{x}{1+x^2}\right)'$$

$$= \frac{1+x^2}{x\ln 10}\cdot\frac{x'(1+x^2)-x(1+x^2)'}{(1+x^2)^2}$$

$$= \frac{1+x^2}{x\ln 10}\cdot\frac{1\cdot(1+x^2)-x\cdot 2x}{(1+x^2)^2}$$

$$= \frac{1+x^2}{x\ln 10}\cdot\frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \frac{1+x^2}{x\ln 10}\cdot\frac{1-x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{x(1+x^2)\ln 10}$$

or

$$f'(x) = \left[\log\left(\frac{x}{1+x^2}\right)\right]' = \left[\log x - \log(1+x^2)\right]'$$

$$= [\log x]' - \left[\log(1+x^2)\right]'$$

$$= \frac{1}{x\ln 10} - \frac{1}{(1+x^2)\ln 10}\cdot(1+x^2)'$$

$$= \frac{1}{x\ln 10} - \frac{2x}{(1+x^2)\ln 10}$$

$$= \frac{1+x^2}{x(1+x^2)\ln 10} - \frac{2x^2}{x(1+x^2)\ln 10}$$

$$= \frac{1+x^2-2x^2}{x(1+x^2)\ln 10}$$

$$= \frac{1-x^2}{x(1+x^2)\ln 10}$$

Applications

EXAMPLE: If a person borrows \$120,000 on a 30-year fixed-rate mortgage at 5.7% interest per year, compounded monthly, then the required monthly payment is \$696. However, the borrower would like to pay off the loan early by making larger monthly payments. The actual number of months it will take to pay off the mortgage if a payment of p dollars is made each month is approximated by

$$f(p) = 211 \ln\left(\frac{p}{p-570}\right) \qquad (p \geq 696) \qquad\qquad \text{[see Appendix]}$$

Find $f(734)$ and $f'(734)$ and explain what your answers mean for the borrower.

Solution: We first note that

$$f(696) = 211 \ln\left(\frac{696}{696-570}\right) \approx 360 \quad \text{and} \quad f(120,000) = 211 \ln\left(\frac{120,000}{120,000-570}\right) \approx 1$$

so the formula makes perfect sense. Evaluating $f(p)$ at $p = 734$ yields:

$$f(734) = 211 \ln\left(\frac{734}{734-570}\right) = 211 \ln\left(\frac{734}{164}\right) \approx 316.2$$

Making monthly payments of \$734 means that the borrower will pay off the loan in 317 months instead of 360 months (the payment in the final month will be somewhat smaller).

Prior to taking the derivative, we may choose to use properties of logarithms to rewrite $f(p)$ as follows:

$$f(p) = 211 \ln\left(\frac{p}{p-570}\right) = 211[\ln p - \ln(p-570)] = 211 \ln(p) - 211 \ln(p-570)$$

Differentiating $f(p)$ with respect to p yields

$$\begin{aligned}
f'(p) = \Big(211 \ln(p) - 211 \ln(p-570)\Big)' &= \Big(211 \ln(p)\Big)' - \Big(211 \ln(p-570)\Big)' \\
&= 211\Big(\ln(p)\Big)' - 211\Big(\ln(p-570)\Big)' \\
&= 211\left(\frac{1}{p}\right) - 211\left(\frac{1}{p-570}\right) \cdot (p-570)' \\
&= 211\left(\frac{1}{p}\right) - 211\left(\frac{1}{p-570}\right) \cdot (1-0) \\
&= 211\left(\frac{1}{p}\right) - 211\left(\frac{1}{p-570}\right) \\
&= \frac{211}{p} - \frac{211}{p-570}
\end{aligned}$$

By evaluating the derivative at $p = 734$ we find that

$$f'(734) = \frac{211}{734} - \frac{211}{734-570} = \frac{211}{734} - \frac{211}{164} \approx -1$$

This means that, when $p = \$734$, increasing the monthly payment by 1 dollar will shorten the length of time needed to pay off the loan by approximately 1 month. (One can check that indeed $f(735) \approx 315.2$.)

Often, a population or the sales of a certain product will start growing slowly, then grow more rapidly, and then gradually level off. Such growth can frequently be approximated by a *logistic function* of the form

$$f(x)=\frac{c}{1+ae^{kx}}$$

for appropriate constants a, c, and k.

EXAMPLE: The number of cell phone subscriptions in the United States (in millions) in year t can be approximated by

$$f(t)=\frac{320}{1+34e^{-.28t}}$$

where $t=0$ corresponds to the year 1990. Find the rate of change of cell phone subscriptions in 1990, 2004, and 2014. (Data from: www.worldbank.org.)

Solution: Use the quotient rule to find the derivative of $f(t)$:

$$\begin{aligned}
f'(t)=\left(\frac{320}{1+34e^{-.28t}}\right)' &= \frac{320'(1+34e^{-.28t})-320(1+34e^{-.28t})'}{(1+34e^{-.28t})^2}\\
&= \frac{(0)(1+34e^{-.28t})-320(1'+(34e^{-.28t})')}{(1+34e^{-.28t})^2}\\
&= \frac{-320(0+34(e^{-.28t})')}{(1+34e^{-.28t})^2}\\
&= \frac{-320(34)(e^{-.28t})'}{(1+34e^{-.28t})^2}\\
&= \frac{-320(34)(e^{-.28t})(-.28t)'}{(1+34e^{-.28t})^2}\\
&= \frac{-320(34)(e^{-.28t})(-.28)}{(1+34e^{-.28t})^2}\\
&= \frac{3046.4e^{-.28t}}{(1+34e^{-.28t})^2}
\end{aligned}$$

The rate of change in 1990 ($t=0$) is

$$f'(0)=\frac{3046.4e^{-.28(0)}}{(1+34e^{-.28(0)})^2}\approx 2.49$$

Similarly, we compute that $f'(14)\approx 21.55$ and $f'(24)\approx 3.39$. This means that, in 1990, cell phone subscriptions were increasing at a rate of 2.49 million per year. By 2004, the rate of change had jumped dramatically to 21.55 million per year. By 2014, the rate of increase had slowed to 3.39 million per year.

Appendix

EXAMPLE: A person borrows \$120,000 on a 30-year fixed-rate mortgage at 5.7% interest per year, compounded monthly.

(a) Find the monthly payment needed to amortize this loan.

Solution: We apply the Amortization Payments formula from Section 5.4 with $n = 12(30) = 360$ (the number of monthly payments in 30 years), and monthly interest rate $i = .057/12$.

$$R = \frac{Pi}{1-(1+i)^{-n}} = \frac{(120,000)(.057/12)}{1-(1+.057/12)^{-360}} = \$696.48$$

Monthly payments of \$696.48 are required to amortize the loan.

(b) The borrower would like to pay off the loan early by making larger monthly payments P. Find the actual number of months n it will take to pay off the mortgage.

Solution: We first solve the Amortization Payments formula for n:

$$R = \frac{Pi}{1-(1+i)^{-n}} \quad \Longleftrightarrow \quad R(1-(1+i)^{-n}) = Pi \quad \Longleftrightarrow \quad R - R(1+i)^{-n} = Pi$$

$$R = Pi + R(1+i)^{-n}$$

$$R - Pi = R(1+i)^{-n}$$

$$\frac{R-Pi}{R} = (1+i)^{-n}$$

$$\ln\left(\frac{R-Pi}{R}\right) = \ln((1+i)^{-n})$$

$$\ln\left(\frac{R-Pi}{R}\right) = -n\ln(1+i)$$

$$\frac{\ln\left(\frac{R-Pi}{R}\right)}{\ln(1+i)} = -n$$

so

$$n = -\frac{\ln\left(\frac{R-Pi}{R}\right)}{\ln(1+i)} = \frac{(-1)\ln\left(\frac{R-Pi}{R}\right)}{\ln(1+i)} = \frac{\ln\left(\frac{R-Pi}{R}\right)^{-1}}{\ln(1+i)} = \frac{\ln\left(\frac{R}{R-Pi}\right)}{\ln(1+i)}$$

Since $P = 120,000$ and $i = 0.057/12$, we get

$$n = \frac{\ln\left(\frac{R}{R-(120,000)(0.057/12)}\right)}{\ln(1+0.057/12)} = \frac{1}{\ln(1+0.057/12)}\ln\left(\frac{R}{R-(120,000)(0.057/12)}\right)$$

$$\approx 211\ln\left(\frac{R}{R-570}\right)$$