Section 11.7 The Chain Rule

Composition of Functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2+1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

> **Composite Function**
>
> Let f and g be functions. The **composite function,** or **composition,** of g and f is the function whose values are given by $g[f(x)]$ for all x in the domain of f such that $f(x)$ is in the domain of g.

EXAMPLE: If $f(x) = x^2+1$ and $g(x) = x-3$, then

(a) $f(f(x)) = \left\{\begin{array}{l} f(x^2+1) \\ \text{or} \\ (f(x))^2+1 \end{array}\right\} = (x^2+1)^2+1 = (x^2)^2 + 2\cdot x^2 \cdot 1 + 1^2 + 1 = x^4+2x^2+2$

(b) $f(g(x)) = \left\{\begin{array}{l} f(x-3) \\ \text{or} \\ (g(x))^2+1 \end{array}\right\} = (x-3)^2+1 = x^2 - 2\cdot x\cdot 3 + 3^2 + 1 = x^2-6x+10$

(c) $g(f(x)) = \left\{\begin{array}{l} g(x^2+1) \\ \text{or} \\ f(x)-3 \end{array}\right\} = (x^2+1)-3 = x^2-2$

(d) $g(g(x)) = \left\{\begin{array}{l} g(x-3) \\ \text{or} \\ g(x)-3 \end{array}\right\} = (x-3)-3 = x-6$

(e) $f(g(2)) = (2-3)^2+1 = (-1)^2+1 = 1+1 = 2, \quad g(f(2)) = 2^2-2 = 4-2 = 2$

EXAMPLE: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, then

(a) $f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$

(b) $g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$

(c) $f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$

(d) $g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$

EXAMPLE: If $f(x) = x$ and $g(x) = 1$, then

$$f(f(x)) = x \qquad f(g(x)) = 1 \qquad g(f(x)) = 1 \qquad g(g(x)) = 1$$

EXAMPLE:

(a) Express the function $h(x) = (x^3 + x^2 - 5)^4$ as the composite of two functions.

Solution: One way to do this is to let $f(x) = x^3 + x^2 - 5$ and $g(x) = x^4$; then

$$g[f(x)] = g[x^3 + x^2 - 5] = (x^3 + x^2 - 5)^4 = h(x).$$

(b) Express the function $h(x) = \sqrt{4x^2 + 5}$ as the composite of two functions in two different ways.

Solution: One way is to let $f(x) = 4x^2 + 5$ and $g(x) = \sqrt{x}$, so that

$$g[f(x)] = g[4x^2 + 5] = \sqrt{4x^2 + 5} = h(x)$$

Another way is to let $k(x) = 4x^2$ and $t(x) = \sqrt{x + 5}$; then

$$t[k(x)] = t[4x^2] = \sqrt{4x^2 + 5} = h(x)$$

The Chain Rule

PROBLEM: Let $f(x) = (1 + x)^2$. Find $f'(x)$.

Solution 1: To find the derivative of this function, we do algebra first and then apply calculus rules:

$$f'(x) = [(1 + x)^2]' = (1 + 2x + x^2)' = 1' + 2(x)' + (x^2)' = 0 + 2 \cdot 1 + 2x = 2 + 2x$$

Solution 2(?): One can try to use the power rule immediately:

$$f'(x) = [(1 + x)^2]' = 2(1 + x)^{2-1} = 2(1 + x)$$

Note that in both cases we got the same result. However, the goal of this Section is to show that despite the fact that Solution 2 gives the right answer, it is not completely correct. To explain what me mean by that, let us consider the following example:

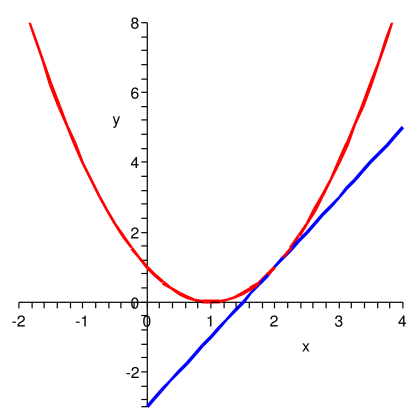
PROBLEM: Let $f(x) = (1 - x)^2$. Find $f'(x)$.

Solution 1: We have

$$\begin{aligned} f'(x) &= [(1 - x)^2]' = (1 - 2x + x^2)' = 1' - 2(x)' + (x^2)' \\ &= 0 - 2 \cdot 1 + 2x = -2 + 2x \end{aligned}$$

Solution 2(???): If we apply the power rule immediately, we get

$$f'(x) = [(1 - x)^2]' \stackrel{?}{=} 2(1 - x)^{2-1} = 2(1 - x)$$



Note that we got two different answers. One can easily see that the second answer is incorrect. In fact, if $f'(x) = 2(1 - x)$, then $f'(2) = 2(1 - 2) = 2(-1) = -2$. This means that the slope of the tangent line to the curve $f(x) = (1 - x)^2$ at $x = 2$ is negative. But this is not the case!

CONCLUSION: We can't always apply the rule $(x^n)' = nx^{n-1}$ to cases when we have u instead of x, where u is an algebraic expression different from x.

THE CHAIN RULE: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

EXAMPLE: If $F(x) = (1-x)^2$, then

$$\begin{aligned} F'(x) = [(1-x)^2]' = [F = f(g(x)) \text{ where } f(x) = x^2, g(x) = 1-x] \; &= 2(1-x) \cdot (1-x)' \\ &= 2(1-x)(-1) \\ &= -2(1-x) \end{aligned}$$

or

$$\frac{d((1-x)^2)}{dx} = [y = u^2, u = 1-x] = \frac{d(u^2)}{du}\frac{d(1-x)}{dx} = 2u \cdot (-1) = 2(1-x)(-1) = -2(1-x)$$

EXAMPLES:

1. $[(3x^2-5x+1)^{50}]' = 50(3x^2-5x+1)^{50-1} \cdot (3x^2-5x+1)' = 50(3x^2-5x+1)^{49}(6x-5)$

2. $[\sqrt[3]{1-4x^2}]' = \dfrac{1}{3}(1-4x^2)^{1/3-1} \cdot (1-4x^2)' = \dfrac{1}{3}(1-4x^2)^{-2/3} \cdot (-8x) = -\dfrac{8}{3}x(1-4x^2)^{-2/3}$

3. $$\begin{aligned} [5x+7(1+2x)^{10}]' = 5x' + 7[(1+2x)^{10}]' &= 5 \cdot 1 + 7 \cdot 10(1+2x)^{10-1} \cdot (1+2x)' \\ &= 5 + 70(1+2x)^9 \cdot 2 \\ &= 5 + 140(1+2x)^9 \end{aligned}$$

4. $[x(x^2-x+1)^{23}]' \;\; =$

5. $\left[\dfrac{1}{x^3+2x-3}\right]' =$

4. $$\begin{aligned}[x(x^2-x+1)^{23}]' &= x'(x^2-x+1)^{23} + x[(x^2-x+1)^{23}]' \\ &= 1\cdot(x^2-x+1)^{23} + x\cdot 23(x^2-x+1)^{23-1}\cdot(x^2-x+1)' \\ &= (x^2-x+1)^{23} + 23x(x^2-x+1)^{22}\cdot((x^2)'-(x)'+(1)') \\ &= (x^2-x+1)^{23} + 23x(x^2-x+1)^{22}\cdot(2x-1+0) \\ &= (x^2-x+1)^{23} + 23x(x^2-x+1)^{22}(2x-1)\end{aligned}$$

5. $$\begin{aligned}\left[\frac{1}{x^3+2x-3}\right]' = [(x^3+2x-3)^{-1}]' &= (-1)(x^3+2x-3)^{-1-1}\cdot(x^3+2x-3)' \\ &= -(x^3+2x-3)^{-2}\cdot((x^3)'+(2x)'-(3)') \\ &= -(x^3+2x-3)^{-2}\cdot((x^3)'+2(x)'-(3)') \\ &= -(x^3+2x-3)^{-2}\cdot(3x^2+2\cdot 1-0) \\ &= -(x^3+2x-3)^{-2}(3x^2+2)\end{aligned}$$

COMMON MISTAKES

1. $[(1-x)^3]' = 3(1-x)^2$ **WRONG!!!**

   Solution: By the Chain Rule we have:

$$\begin{aligned}[(1-x)^3]' &= 3(1-x)^{3-1}\cdot(1-x)' \\ &= 3(1-x)^2\cdot((1)'-(x)') \\ &= 3(1-x)^2\cdot(0-1) \\ &= 3(1-x)^2\cdot(-1) \\ &= -3(1-x)^2\end{aligned}$$

2. $[(x+x^5)^4]' = 4(1+5x^4)^3$ **WRONG!!!**

   Solution: By the Chain Rule we have

$$\begin{aligned}[(x+x^5)^4]' &= 4(x+x^5)^{4-1}\cdot(x+x^5)' \\ &= 4(x+x^5)^3\cdot((x)'+(x^5)') \\ &= 4(x+x^5)^3(1+5x^4)\end{aligned}$$

Applications

EXAMPLE: During a week-long promotion, the profit generated by an online sporting goods retailer from the sale of n official basketballs is given by

$$P(n) = \frac{45n^2}{3n+10}$$

Sales are approximately constant at a rate of 25 basketballs per day, therefore

$$\frac{dn}{dt} = 25$$

How fast is profit changing 4 days after the start of the promotion?

Solution: We want to find $\frac{dP}{dt}$, the rate of change of profit with respect to time. By the chain rule,

$$\frac{dP}{dt} = \frac{dP}{dn}\frac{dn}{dt}$$

First find $\frac{dP}{dn}$ as follows:

$$\begin{aligned}\frac{dP}{dn} &= \frac{(45n^2)'(3n+10) - (45n^2)(3n+10)'}{(3n+10)^2}\\ &= \frac{45(n^2)'(3n+10) - (45n^2)((3n)' + (10)')}{(3n+10)^2}\\ &= \frac{45(2n)(3n+10) - (45n^2)(3+0)}{(3n+10)^2}\\ &= \frac{(90n)(3n+10) - (45n^2)(3)}{(3n+10)^2}\\ &= \frac{270n^2 + 900n - 135n^2}{(3n+10)^2}\\ &= \frac{135n^2 + 900n}{(3n+10)^2}\end{aligned}$$

With sales at 25 basketballs per day, 4 days after the start of the promotion,

$$n = \text{ (4 days)(25 basketballs/day)} = 100 \text{ basketballs}$$

So after 4 days,

$$\frac{dP}{dn} = \frac{135(100)^2 + 900(100)}{(3(100)+10)^2} \approx 14.9844$$

We are given that $\frac{dn}{dt} = 25$, so we have

$$\frac{dP}{dt} = \frac{dP}{dn}\frac{dn}{dt} \approx (14.9844)(25) \approx 374.61$$

After 4 days, profit from basketballs is increasing at a rate of \$374.61 per day.

EXAMPLE: A generous aunt deposits $20,000 in an account to be used by her newly born niece to attend college. The account earns interest at the rate of r percent per year, compounded monthly. At the end of 18 years, the balance in the account is given by

$$A = 20,000\left(1+\frac{r}{1200}\right)^{216}$$

Find the rate of change of A with respect to r if $r = 1.5,\ 2.5$, or 3.

Solution: First find dA/dr, using the generalized power rule:

$$\begin{aligned}
\frac{dA}{dr} = \left(20,000\left(1+\frac{r}{1200}\right)^{216}\right)' &= 20,000\left(\left(1+\frac{r}{1200}\right)^{216}\right)' \\
&= 20,000(216)\left(1+\frac{r}{1200}\right)^{216-1}\cdot\left(1+\frac{r}{1200}\right)' \\
&= 20,000(216)\left(1+\frac{r}{1200}\right)^{215}\cdot\left((1)'+\left(\frac{r}{1200}\right)'\right) \\
&= 20,000(216)\left(1+\frac{r}{1200}\right)^{215}\cdot\left((1)'+\left(\frac{1}{1200}r\right)'\right) \\
&= 20,000(216)\left(1+\frac{r}{1200}\right)^{215}\cdot\left((1)'+\frac{1}{1200}(r)'\right) \\
&= 20,000(216)\left(1+\frac{r}{1200}\right)^{215}\cdot\left(0+\frac{1}{1200}(1)\right) \\
&= 20,000(216)\left(1+\frac{r}{1200}\right)^{215}\frac{1}{1200} \\
&= 3600\left(1+\frac{r}{1200}\right)^{215}
\end{aligned}$$

If $r = 1.5$, we obtain

$$\frac{dA}{dr} = 3600\left(1+\frac{1.5}{1200}\right)^{215} = 4709.19$$

or $4709.19 per percentage point. If $r = 2.5$, we obtain

$$\frac{dA}{dr} = 3600\left(1+\frac{2.5}{1200}\right)^{215} = 5631.55$$

or $5631.55 per percentage point. If $r = 3$, we obtain

$$\frac{dA}{dr} = 3600\left(1+\frac{3}{1200}\right)^{215} = 6158.07$$

or $6158.07 per percentage point. This means that when the interest rate is 3%, an increase of 1% in the interest rate will produce an increase in the balance of approximately $6158.07.

The chain rule can be used to develop the formula for the **marginal-revenue product**, an economic concept that approximates the change in revenue when a manufacturer hires an additional employee. Start with $R = px$, where R is total revenue from the daily production of x units and p is the price per unit. The demand function is $p = f(x)$, as before. Also, x can be considered a function of the number of employees, n. Since $R = px$, and x — and therefore, p — depends on n, R can also be considered a function of n. To find an expression for dR/dn, use the product rule for derivatives on the function $R = px$ to get

$$\frac{dR}{dn} = p \cdot \frac{dx}{dn} + x \cdot \frac{dp}{dn} \qquad (*)$$

By the chain rule,

$$\frac{dp}{dn} = \frac{dp}{dx} \cdot \frac{dx}{dn}$$

Substituting for dp/dn in equation $(*)$ yields

$$\frac{dR}{dn} = p \cdot \frac{dx}{dn} + x\left(\frac{dp}{dx} \cdot \frac{dx}{dn}\right) = \left(p + x \cdot \frac{dp}{dx}\right)\frac{dx}{dn}$$

The expression for dR/dn gives the marginal-revenue product.

EXAMPLE: Find the marginal-revenue product dR/dn (in dollars per employee) when $n = 20$ if the demand function is $p = 600/\sqrt{x}$ and $x = 5n$.

Solution: As shown previously,

$$\frac{dR}{dn} = \left(p + x \cdot \frac{dp}{dx}\right)\frac{dx}{dn}$$

Find dp/dx and dx/dn. From

$$p = \frac{600}{\sqrt{x}} = 600x^{-1/2}$$

we have the derivative

$$\frac{dp}{dx} = (600x^{-1/2})' = 600(x^{-1/2})' = 600(-1/2)x^{-1/2-1} = -300x^{-3/2}$$

Also, from $x = 5n$, we have

$$\frac{dx}{dn} = (5n)' = 5(n)' = 5(1) = 5$$

Then, by substitution,

$$\frac{dR}{dn} = \left(p + x \cdot \frac{dp}{dx}\right)\frac{dx}{dn} = \left[\frac{600}{\sqrt{x}} + x(-300x^{-3/2})\right]5 = \left[\frac{600}{\sqrt{x}} - 300x^{-3/2+1}\right]5$$

$$= \left[\frac{600}{\sqrt{x}} - 300x^{-1/2}\right]5 = \left[\frac{600}{\sqrt{x}} - \frac{300}{\sqrt{x}}\right]5 = \left[\frac{600-300}{\sqrt{x}}\right]5 = \left[\frac{300}{\sqrt{x}}\right]5 = \frac{1500}{\sqrt{x}}$$

If $n = 20$, then $x = 5 \cdot 20 = 100$, and

$$\frac{dR}{dn} = \frac{1500}{\sqrt{100}} = \frac{1500}{10} = 150$$

This means that hiring an additional employee when production is at a level of 100 items will produce an increase in revenue of \$150.