Section 11.6 Derivatives of Products and Quotients

THE PRODUCT RULE: If f and g are both differentiable functions, then

$$\boxed{\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]}$$

or

$$\boxed{[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)} \quad \text{or} \quad \boxed{(fg)' = f'g + fg'}$$

Proof: We have

$$\begin{aligned}
[f(x)g(x)]' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)[f(x+h) - f(x)] + f(x)[g(x+h) - g(x)]}{h} \\
&= \lim_{h\to 0} \left(\frac{g(x+h)[f(x+h) - f(x)]}{h} + \frac{f(x)[g(x+h) - g(x)]}{h} \right) \\
&= \lim_{h\to 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h) - g(x)]}{h} \\
&= \lim_{h\to 0} g(x+h) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x) = f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

EXAMPLE: Let $f(x) = (x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})$. We first note that

$$f'(x) \neq (x^3 + 7x^2 - 8)'(2x^{-3} + x^{-4})'$$

By the Product Rule we have

$$\begin{aligned}
f'(x) &= [(x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})]' \\
&= (x^3 + 7x^2 - 8)'(2x^{-3} + x^{-4}) + (x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})' \\
&= \Big((x^3)' + (7x^2)' - (8)'\Big)(2x^{-3} + x^{-4}) + (x^3 + 7x^2 - 8)\Big((2x^{-3})' + (x^{-4})'\Big) \\
&= \Big((x^3)' + 7(x^2)' - (8)'\Big)(2x^{-3} + x^{-4}) + (x^3 + 7x^2 - 8)\Big(2(x^{-3})' + (x^{-4})'\Big) \\
&= \Big(3x^2 + 7(2x) - 0\Big)(2x^{-3} + x^{-4}) + (x^3 + 7x^2 - 8)\Big(2(-3x^{-4}) + (-4x^{-5})\Big) \\
&= (3x^2 + 14x)(2x^{-3} + x^{-4}) + (x^3 + 7x^2 - 8)(-6x^{-4} - 4x^{-5})
\end{aligned}$$

THE QUOTIENT RULE: If f and g are both differentiable functions, then

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}}$$

or

$$\boxed{\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}} \quad \text{or} \quad \boxed{\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}}$$

EXAMPLE: Find $f'(x)$ if $f(x)=\dfrac{2x^3-4x\sqrt{x}}{x+1}$.

Solution: We first note that

$$f'(x)\neq\frac{(2x^3-4x\sqrt{x})'}{(x+1)'}$$

To find f' we first simplify the expression:

$$f(x)=\frac{2x^3-4x\sqrt{x}}{x+1}=\frac{2x^3-4x^1\cdot x^{1/2}}{x+1}=\frac{2x^3-4x^{1+1/2}}{x+1}=\frac{2x^3-4x^{3/2}}{x+1}$$

We now apply the Quotient Rule:

$$\begin{aligned}
f'(x)=\left(\frac{2x^3-4x^{3/2}}{x+1}\right)' &= \frac{(2x^3-4x^{3/2})'(x+1)-(2x^3-4x^{3/2})(x+1)'}{(x+1)^2}\\
&=\frac{\Big((2x^3)'-(4x^{3/2})'\Big)(x+1)-(2x^3-4x^{3/2})(x'+1')}{(x+1)^2}\\
&=\frac{\left(2(3x^2)-4\cdot\dfrac{3}{2}x^{1/2}\right)(x+1)-(2x^3-4x^{3/2})(1+0)}{(x+1)^2}\\
&=\frac{(6x^2-6x^{1/2})(x+1)-(2x^3-4x^{3/2})}{(x+1)^2}
\end{aligned}$$

which can be rewritten as $\dfrac{4x^3+6x^2-2x^{3/2}-6x^{1/2}}{(x+1)^2}$.

EXAMPLE: If $f(x)=\dfrac{2x^3-4x\sqrt{x}}{x}$, then

$$\begin{aligned}
f'(x)&=\left(\frac{2x^3-4x\cdot x^{1/2}}{x}\right)'=\left(\frac{2x^3-4x^{3/2}}{x}\right)'=\frac{(2x^3-4x^{3/2})'x-(2x^3-4x^{3/2})x'}{x^2}\\
&=\frac{\left(2(3x^2)-4\cdot\dfrac{3}{2}x^{1/2}\right)x-(2x^3-4x^{3/2})\cdot 1}{x^2}=\frac{(6x^2-6x^{1/2})x-(2x^3-4x^{3/2})}{x^2}\\
&=\frac{6x^3-6x^{3/2}-2x^3+4x^{3/2}}{x^2}=\frac{4x^3-2x^{3/2}}{x^2}=4x-2x^{-1/2}
\end{aligned}$$

or

$$f'(x)=\left(\frac{2x^3-4x\cdot x^{1/2}}{x}\right)'=\left(\frac{2x^3}{x}-\frac{4x\cdot x^{1/2}}{x}\right)'=\left(2x^2-4x^{1/2}\right)'=2(x^2)'-4(x^{1/2})'=4x-2x^{-1/2}$$

EXAMPLE: Find $f'(x)$ if $f(x)=\dfrac{1}{x}$.

Solution:

$$f'(x)=\left(\frac{1}{x}\right)'=\frac{1'\cdot x-1\cdot x'}{x^2}=\frac{0\cdot x-1\cdot 1}{x^2}=\frac{0-1}{x^2}=\frac{-1}{x^2}=-\frac{1}{x^2}$$

or

$$f'(x)=\left(\frac{1}{x}\right)'=(x^{-1})'=(-1)x^{-1-1}=-x^{-2}$$

EXAMPLES:

1. $(3x-5)'=(3x)'-5'=3x'-5'=3\cdot 1-0=3$

2. $(\sqrt{x})'=\left(x^{1/2}\right)'=\dfrac{1}{2}x^{1/2-1}=\dfrac{1}{2}x^{-1/2}$

3. $\left(\dfrac{1}{\sqrt{x}}\right)'=\left(\dfrac{1}{x^{1/2}}\right)'=\left(x^{-1/2}\right)'=-\dfrac{1}{2}x^{-1/2-1}=-\dfrac{1}{2}x^{-3/2}$

4. $(-3x^{-8}+2\sqrt{x})'=-3(x^{-8})'+2(x^{1/2})'\ =-3(-8)x^{-8-1}+2\dfrac{1}{2}x^{-1/2}=24x^{-9}+x^{-1/2}$

5. $$\left[\left(\frac{3x+2}{x+1}\right)(x^{-5}+1)\right]'=\left(\frac{3x+2}{x+1}\right)'(x^{-5}+1)+\left(\frac{3x+2}{x+1}\right)(x^{-5}+1)'$$
$$=\frac{(3x+2)'(x+1)-(3x+2)(x+1)'}{(x+1)^2}(x^{-5}+1)+\left(\frac{3x+2}{x+1}\right)(x^{-5}+1)'$$
$$=\frac{3(x+1)-(3x+2)}{(x+1)^2}(x^{-5}+1)+\left(\frac{3x+2}{x+1}\right)(-5x^{-6})=\frac{x^{-5}+1}{(x+1)^2}+\left(\frac{3x+2}{x+1}\right)(-5x^{-6})$$

Average Cost

Suppose $y=C(x)$ gives the total cost of manufacturing x items. As mentioned earlier, the average cost per item is found by dividing the total cost by the number of items. The rate of change of average cost, called the *marginal average cost*, is the derivative of the average cost.

Average Cost

If the total cost of manufacturing x items is given by $C(x)$, then the **average cost per item** is

$$\overline{C}(x)=\frac{C(x)}{x}.$$

The **marginal average cost** is the derivative of the average-cost function $\overline{C}'(x)$.

A company naturally would be interested in making the average cost as small as possible. We will see in the next chapter that this can be done by using the derivative of $C(x)/x$. The derivative often can be found with the quotient rule, as in the next example.

EXAMPLE: The total cost (in dollars) to manufacture x mobile phones is given by

$$C(x)=\frac{50x^2+30x+4}{x+2}+80,000$$

(a) Find the average cost per phone.

Solution: The average cost is given by the total cost divided by the number of items:

$$\begin{aligned}\overline{C}(x)=\frac{C(x)}{x}=\frac{1}{x}C(x)&=\frac{1}{x}\left(\frac{50x^2+30x+4}{x+2}+80,000\right)\\&=\frac{1}{x}\cdot\frac{50x^2+30x+4}{x+2}+\frac{1}{x}\cdot 80,000\\&=\frac{50x^2+30x+4}{x(x+2)}+x^{-1}\cdot 80,000\\&=\frac{50x^2+30x+4}{x^2+2x}+80,000x^{-1}\end{aligned}$$

(b) Find the average cost per phone for each of the following production levels:

5000; 10,000; 100,000.

Solution: Evaluate $C(x)$ at each of the numbers, either by hand or by using technology:

$$\overline{C}(5000)=65,986;\quad \overline{C}(10,000)=57,993;\quad \overline{C}(100,000)=50,799.$$

Note that the average cost per phone is \$65.99 when 5000 phones are produced, and that reduces to \$50.80 when 100,000 are produced.

(c) Find the marginal average cost.

Solution: The marginal average cost is the derivative of the average-cost function. Using the sum rule and the quotient rule yields

$$\begin{aligned}\overline{C}'(x)&=\left(\frac{50x^2+30x+4}{x^2+2x}+80,000x^{-1}\right)'\\&=\left(\frac{50x^2+30x+4}{x^2+2x}\right)'+(80,000x^{-1})'\\&=\frac{(50x^2+30x+4)'(x^2+2x)-(50x^2+30x+4)(x^2+2x)'}{(x^2+2x)^2}+80,000(x^{-1})'\\&=\frac{(100x+30)(x^2+2x)-(50x^2+30x+4)(2x+2)}{(x^2+2x)^2}+(-1)80,000x^{-2}\\&=\frac{(100x^3+230x^2+60x)-(100x^3+160x^2+68x+8)}{(x^2+2x)^2}-\frac{80,000}{x^2}\\&=\frac{70x^2-8x-8}{(x^2+2x)^2}-\frac{80,000}{x^2}\end{aligned}$$

(d) Find the marginal average cost at production levels of 500 phones and 1000 phones.

Solution: Evaluating the marginal average-cost function at $x = 500$ yields

$$\overline{C}'(500) = \frac{70(500)^2 - 8(500) - 8}{(500^2 + 2(500))^2} - \frac{80,000}{500^2} \approx -.32$$

Therefore, at a production level of 500 phones, if an additional phone is produced, the *average cost per phone* is decreased by approximately 32 cents per phone.

Similarly, $\overline{C}'(1000) \approx -.08$ means that, at a production level of 1000 phones, if an additional phone is produced, the average cost per phone is decreased by approximately 8 cents per phone.

EXAMPLE: The cost (in hundreds of dollars) incurred by a flower shop to produce x hundred flower arrangements for delivery on Valentine's day is given by

$$C(x) = x^2 + 3x + 18$$

(a) Find the marginal cost at a production level of 200 flower arrangements.

Solution: Since the marginal cost function is

$$C'(x) = (x^2 + 3x + 18)' = (x^2)' + (3x)' + (18)' = 2x + 3$$

the marginal cost at a production level of 200 arrangements is $C'(2) = 2(2) + 3 = 7$. At this production level, the cost of producing an *additional* hundred arrangements is approximately \$700.

(b) Find the average cost at a production level of 200 flower arrangements.

Solution: The average cost is

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{x^2 + 3x + 18}{x} = \frac{x^2}{x} + \frac{3x}{x} + \frac{18}{x} = x + 3 + \frac{18}{x}$$

Evaluating at $x = 2$ yields $\overline{C}(2) = 2 + 3 + \dfrac{18}{2} = 14$. At a production level of 200 arrangements, the average cost is \$1400 per hundred arrangements (or \$14 per arrangement).

(c) Find the marginal average cost at production levels of 200 and 500 flower arrangements.

Solution: The marginal average cost is the derivative of the average-cost function:

$$\begin{aligned}\overline{C}'(x) = \left(x + 3 + \frac{18}{x}\right)' &= x' + 3' + (18x^{-1})' \\ &= x' + 3' + 18(x^{-1})' \\ &= 1 + 0 + 18(-1)x^{-1-1} = 1 - 18x^{-2} = 1 - \frac{18}{x^2}\end{aligned}$$

Evaluating at $x = 2$ yields

$$\overline{C}'(2) = 1 - \frac{18}{2^2} = -3.5$$

At this production level, if an additional hundred arrangements are produced, the average cost will be *decreased* by approximately \$350 per hundred arrangements. Increasing production will lower the average cost.

Similarly,

$$\overline{C}'(5) = 1 - \frac{18}{5^2} = .28$$

At this higher production level, if an additional hundred arrangements are produced, the average cost will be *increased* by approximately \$28 per hundred arrangements. It would not make good business sense to increase production, since it would raise the average cost.

(d) Find the level of production at which the marginal average cost is zero.

Solution: Set the derivative $\overline{C}'(x) = 0$ and solve for x:

$$\begin{aligned} 1 - \frac{18}{x^2} &= 0 \\ 1 &= \frac{18}{x^2} \\ x^2 &= 18 \\ x &= \pm\sqrt{18} \approx \pm 4.24 \end{aligned}$$

You cannot make a negative number of flower arrangements, so $x = 4.24$. The production of 424 flower arrangements will yield a marginal average cost of zero dollars.