Section 11.5 Techniques for Finding Derivatives

In the previous section, the derivative of a function was defined as a special limit. The mathematical process of finding this limit, called *differentiation*, resulted in a new function that was interpreted in several different ways. Using the definition to calculate the derivative of a function is a very involved process, even for simple functions. In this section, we develop rules that make the calculation of derivatives much easier. Keep in mind that even though the process of finding a derivative will be greatly simplified with these rules, *the interpretation of the derivative will not change.*

In addition to y' and $f'(x)$, there are several other commonly used notations for the derivative.

Notations for the Derivative

The derivative of the function $y = f(x)$ may be denoted in any of the following ways:

$$f'(x), \qquad y', \qquad \frac{dy}{dx}, \qquad \frac{d}{dx}[f(x)], \qquad D_x y, \qquad \text{or} \qquad D_x[f(x)].$$

DERIVATIVE OF A CONSTANT FUNCTION:

$$\boxed{\frac{d}{dx}(c) = 0} \quad \text{or} \quad \boxed{c' = 0}$$

Proof: Suppose $f(x) = c$, then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

EXAMPLES:

$$1' = 0, \qquad 5' = 0, \qquad 0' = 0, \qquad (-7/9)' = 0, \qquad \pi' = 0, \qquad \left(\frac{1+\sqrt{5}}{2}\right)' = 0$$

THE POWER RULE: If n is a real number, then

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}} \quad \text{or} \quad \boxed{(x^n)' = nx^{n-1}}$$

Proof: Here we prove this result for $n = 2$ and $n = 3$ only.

If $n = 2$, then

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x-a)(x+a)}{x-a} = \lim_{x\to a}(x+a) = a+a = 2a$$

If $n = 3$, then

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^3-a^3}{x-a} = \lim_{x\to a}\frac{(x-a)(x^2+xa+a^2)}{x-a}$$

$$= \lim_{x\to a}(x^2+xa+a^2) = a^2 + a\cdot a + a^2 = 3a^2$$

EXAMPLES:

(a) If $f(x)=x^2$, then $f'(x)=(x^2)'=[n=2]=2x^{2-1}=2x^1=2x$.

(b) If $f(x)=x^9$, then $f'(x)=(x^9)'=[n=9]=9x^{9-1}=9x^8$.

(c) If $f(x)=x$, then $f'(x)=(x^1)'=[n=1]=1\cdot x^{1-1}=1\cdot x^0=1\cdot 1=1$.

(d) If $f(x)=x^{-4}$, then $f'(x)=(x^{-4})'=[n=-4]=(-4)x^{-4-1}=-4x^{-5}$.

(e) If $f(x)=\dfrac{1}{x}$, then $f'(x)=(x^{-1})'=[n=-1]=(-1)x^{-1-1}=-x^{-2}=-\dfrac{1}{x^2}$.

(f) If $f(x)=\sqrt{x}$, then $f'(x)=(x^{1/2})'=[n=1/2]=\dfrac{1}{2}x^{1/2-1}=\dfrac{1}{2}x^{-1/2}=\dfrac{1}{2\sqrt{x}}$.

(g) If $f(x)=x^2\sqrt[3]{x}$, then

$$f'(x)=(x^2\cdot x^{1/3})'=(x^{2+1/3})'=(x^{7/3})'=[n=7/3]=\frac{7}{3}x^{7/3-1}=\frac{7}{3}x^{4/3}$$

(h) If $f(x)=\dfrac{1}{\sqrt[3]{x^2}}$, then

$$f'(x)=\left(\frac{1}{x^{2/3}}\right)'=(x^{-2/3})'=[n=-2/3]=-\frac{2}{3}x^{-2/3-1}=-\frac{2}{3}x^{-5/3}$$

(i) If $f(x)=\dfrac{\sqrt[4]{x}}{x^{-1}\sqrt{x^5}}$, then

$$\begin{aligned}
f'(x)=\left(\frac{x^{1/4}}{x^{-1}\cdot x^{5/2}}\right)' &= \left(\frac{x^{1/4}}{x^{-1+5/2}}\right)' \\
&= \left(x^{1/4-(-1+5/2)}\right)' \\
&= \left(x^{1/4+1-5/2}\right)' \\
&= \left(x^{-5/4}\right)'=[n=-5/4]=-\frac{5}{4}x^{-5/4-1}=-\frac{5}{4}x^{-9/4}
\end{aligned}$$

THE CONSTANT MULTIPLE RULE: If c is a constant and f is a differentiable function, then

$$\boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)} \quad \text{or} \quad \boxed{(cf(x))' = cf'(x)} \quad \text{or} \quad \boxed{(cf)' = cf'}$$

Proof: We have

$$(cf(x))' = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0}\frac{c[f(x+h)-f(x)]}{h} = c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x)$$

EXAMPLE: If $f(x) = \dfrac{2}{3\sqrt[5]{x}}$, then

$$f'(x) = \left(\frac{2}{3}x^{-1/5}\right)' = \frac{2}{3}\left(x^{-1/5}\right)' = \frac{2}{3}\left(-\frac{1}{5}\right)x^{-1/5-1} = -\frac{2}{15}x^{-6/5}$$

THE SUM/DIFFERENCE RULE: If f and g are both differentiable functions, then

$$\boxed{\frac{d}{dx}[f(x)\pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)}$$

or

$$\boxed{(f(x)\pm g(x))' = f'(x)\pm g'(x)} \quad \text{or} \quad \boxed{(f\pm g)' = f'\pm g'}$$

Proof: We have

$$\begin{aligned}(f(x)\pm g(x))' &= \lim_{h\to 0}\frac{(f(x+h)\pm g(x+h)) - (f(x)\pm g(x))}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]\pm[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)\pm g'(x)\end{aligned}$$

EXAMPLE: If $f(x) = \dfrac{3x^2 - 5\sqrt{x}}{6x^4}$, then

$$\begin{aligned}f'(x) = \left(\frac{3x^2-5x^{1/2}}{6x^4}\right)' = \left(\frac{3x^2}{6x^4} - \frac{5x^{1/2}}{6x^4}\right)' = \left(\frac{3}{6}x^{2-4} - \frac{5}{6}x^{1/2-4}\right)' = \left(\frac{1}{2}x^{-2} - \frac{5}{6}x^{-7/2}\right)'\\ = \left(\frac{1}{2}x^{-2}\right)' - \left(\frac{5}{6}x^{-7/2}\right)' = \frac{1}{2}\left(x^{-2}\right)' - \frac{5}{6}\left(x^{-7/2}\right)'\\ = \frac{1}{2}\cdot(-2)x^{-2-1} - \frac{5}{6}\cdot\left(-\frac{7}{2}\right)\cdot x^{-7/2-1} = -x^{-3} + \frac{35}{12}x^{-9/2}\end{aligned}$$

Marginal Analysis

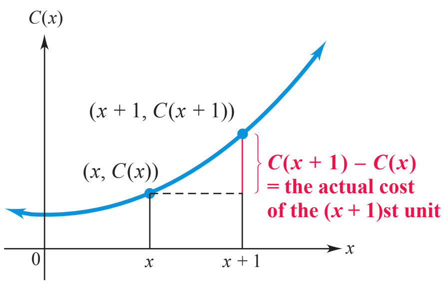
In business and economics, the rates of change of such variables as cost, revenue, and profit are important considerations. Economists use the word *marginal* to refer to rates of change: for example, *marginal cost* refers to the rate of change of cost with respect to the number of items produced. Since the derivative of a function gives the rate of change of the function, a marginal cost (or revenue, or profit) function is found by taking the derivative of the cost (or revenue, or profit) function. Roughly speaking, the marginal cost at some level of production x is the cost of producing the $(x+1)$st item, as we now show. (Similar statements could be made for revenue or profit.)

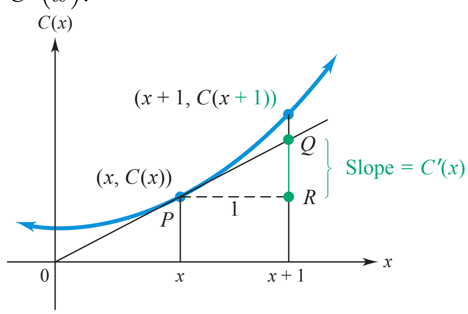
Look at the Figure below (left), where $C(x)$ represents the cost of producing x units of some item. Then the cost of producing $x+1$ units is $C(x+1)$. The cost of the $(x+1)$st unit is, therefore, $C(x+1) - C(x)$.

Now, if C is the cost function, then the marginal cost C' represents the slope of the tangent line at any point $(x, C(x))$. The graph in the Figure below (right) shows the cost function C and the tangent line at point $P = (x, C(x))$. We know that the slope of the tangent line at P is $C'(x)$ and that the slope can be computed using the triangle PQR in the Figure:

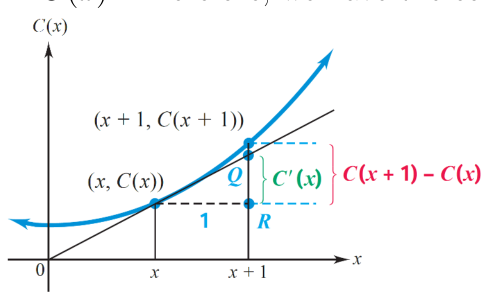
$$C'(x) = \text{ slope } = \frac{QR}{PR} = \frac{QR}{1} = QR$$

So the length of the line segment QR is the number $C'(x)$.





Superimposing the graphs from the Figures above, as in the Figure below, shows that $C'(x)$ is indeed very close to $C(x+1) - C(x)$. Therefore, we have the conclusion in the following box.



Marginal Cost

If $C(x)$ is the cost function, then the marginal cost (rate of change of cost) is given by the derivative $C'(x)$:

$$C'(x) \approx \text{ cost of making one more item after } x \text{ items have been made.}$$

The marginal revenue $R'(x)$ and marginal profit $P'(x)$ are interpreted similarly.

EXAMPLE: A land developer has purchased 115 acres in the Blue Ridge Mountains to subdivide and sell as one-acre lots for mountain homes. Construction of gravel roads to access the lots is more expensive at higher elevations on the property. The cost of access roads (in thousands of dollars) for x lots can be approximated by

$$C(x) = 150 + 9x + .5x^{1.6} \quad (0 \leq x \leq 115)$$

Find and interpret the marginal cost for the given values of x.

(a) $x = 25$

Solution: To find the marginal cost, first find the derivative of the cost function:

$$\begin{aligned} C'(x) &= (150 + 9x + .5x^{1.6})' \\ &= (150)' + (9x)' + (.5x^{1.6})' \\ &= (150)' + 9(x)' + .5(x^{1.6})' \\ &= 0 + 9(1) + .5(1.6x^{1.6-1}) \\ &= 9 + .8x^{.6} \end{aligned}$$

When $x = 25$,

$$C'(25) = 9 + .8(25)^{.6} = 14.519$$

After 25 access roads have been constructed, the cost of constructing one more access road will be *approximately* \$14,519.

Note that the *actual* cost of constructing one more access road is:

$$C(26) - C(25) = (150 + 9(26) + .5(26)^{1.6}) - (150 + 9(25) + .5(25)^{1.6}) \approx 14.585$$

So the actual cost to construct one additional road is \$14,585.

(b) $x = 50$

Solution: After 50 access roads have been constructed, the cost of constructing an additional access road will be *approximately*

$$C'(50) = 9 + .8(50)^{.6} = 17.365$$

or \$17,365. Compare this result with that for part (a). The cost of accessing one additional lot is almost \$3000 more after 50 lots have road access than the cost of accessing one additional lot when only 25 lots have road access. Management must be careful to keep track of marginal costs. If the marginal cost of producing an extra unit exceeds the revenue received from selling it, the company will lose money on that unit.

Demand Functions

The **demand function**, defined by $p = f(x)$, relates the number of units, x, of an item that consumers are willing to purchase at the price p. (Demand functions were also discussed in Section 3.3 .) The total revenue $R(x)$ is related to the price per unit and the amount demanded (or sold) by the equation

$$R(x) = xp = x \cdot f(x)$$

EXAMPLE: The demand function for a certain product is given by

$$p = \frac{50,000 - x}{25,000}$$

Find the marginal revenue when $x = 10,000$ units and p is in dollars.

Solution: From the function p, the revenue function is given by

$$R(x) = xp = x\left(\frac{50,000 - x}{25,000}\right) = \frac{50,000x - x^2}{25,000}$$

$$= \frac{50,000x}{25,000} - \frac{x^2}{25,000} = 2x - \frac{1}{25,000}x^2$$

The marginal revenue is

$$R'(x) = \left(2x - \frac{1}{25,000}x^2\right)'$$

$$= (2x)' - \left(\frac{1}{25,000}x^2\right)'$$

$$= 2(x)' - \frac{1}{25,000}(x^2)'$$

$$= 2(1) - \frac{1}{25,000}(2x)$$

$$= 2 - \frac{2}{25,000}x$$

When $x = 10,000$, the marginal revenue is

$$R'(10,000) = 2 - \frac{2}{25,000}(10,000) = 1.2$$

or \$1.20 per unit. Thus, the next unit sold (at sales of 10,000) will produce additional revenue of about \$1.20.

In economics, the demand function is written in the form $p = f(x)$, as in the Example above. From the perspective of a consumer, it is probably more reasonable to think of the quantity demanded as a function of price. Mathematically, these two viewpoints are equivalent. In the Example above, the demand function could have been written from the consumer's viewpoint as

$$x = 50,000 - 25,000p$$

EXAMPLE: Suppose that the cost function for the product in the Example above is given by

$$C(x) = 2100 + .25x \quad (0 \le x \le 30,000)$$

Find the marginal profit from the production of the given numbers of units.

(a) 15,000

Solution: From the Example above, the revenue from the sale of x units is

$$R(x) = 2x - \frac{1}{25,000}x^2$$

Since profit P is given by $P = R - C$,

$$P(x) = R(x) - C(x) = \left(2x - \frac{1}{25,000}x^2\right) - (2100 + .25x) = 2x - \frac{1}{25,000}x^2 - 2100 - .25x$$

$$= 1.75x - \frac{1}{25,000}x^2 - 2100$$

The marginal profit from the sale of x units is

$$P'(x) = \left(1.75x - \frac{1}{25,000}x^2 - 2100\right)'$$

$$= (1.75x)' - \left(\frac{1}{25,000}x^2\right)' - (2100)'$$

$$= 1.75(x)' - \frac{1}{25,000}(x^2)' - (2100)'$$

$$= 1.75(1) - \frac{1}{25,000}(2x) - 0 = 1.75 - \frac{2}{25,000}x = 1.75 - \frac{1}{12,500}x$$

At $x = 15,000$, the marginal profit is

$$P'(15,000) = 1.75 - \frac{1}{12,500}(15,000) = .55$$

or \$.55 per unit. This means that the next unit sold (at sales of 15,000) will produce additional profit of about 55¢.

(b) 21,875

Solution: When $x = 21,875$, the marginal profit is

$$P'(21,875) = 1.75 - \frac{1}{12,500}(21,875) = 0$$

(c) 25,000

Solution: When $x = 25,000$, the marginal profit is

$$P'(25,000) = 1.75 - \frac{1}{12,500}(25,000) = -.25$$

or $-$\$.25 per unit.

As shown by parts (b) and (c), if more than 21,875 units are sold, the marginal profit is negative. This indicates that increasing production beyond that level will reduce profit.