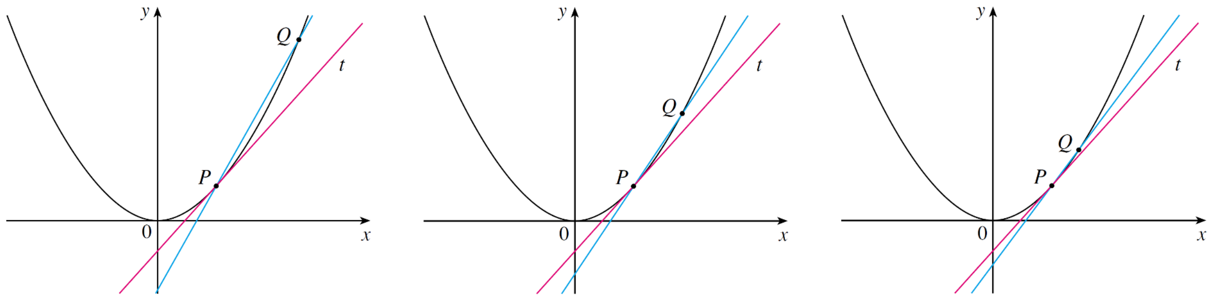
Section 11.4 Tangent Lines and Derivatives

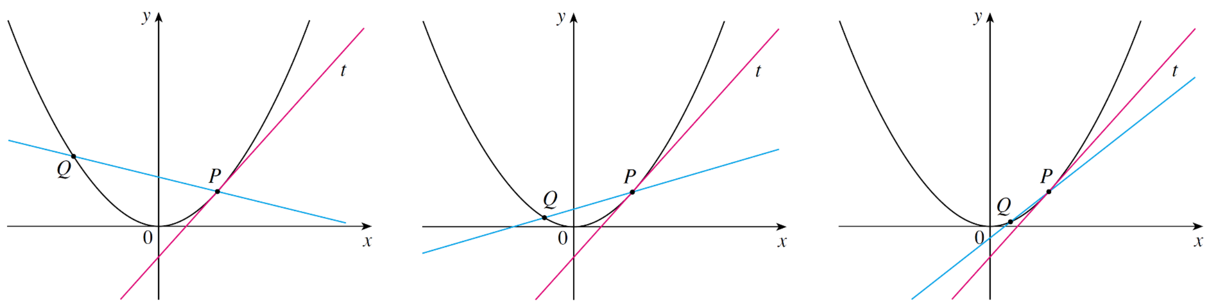
The Tangent Problem

EXAMPLE: Graph the parabola $y = x^2$ and the tangent line at the point $P(1, 1)$.

Solution: We have:



Q approaches P from the right



Q approaches P from the left

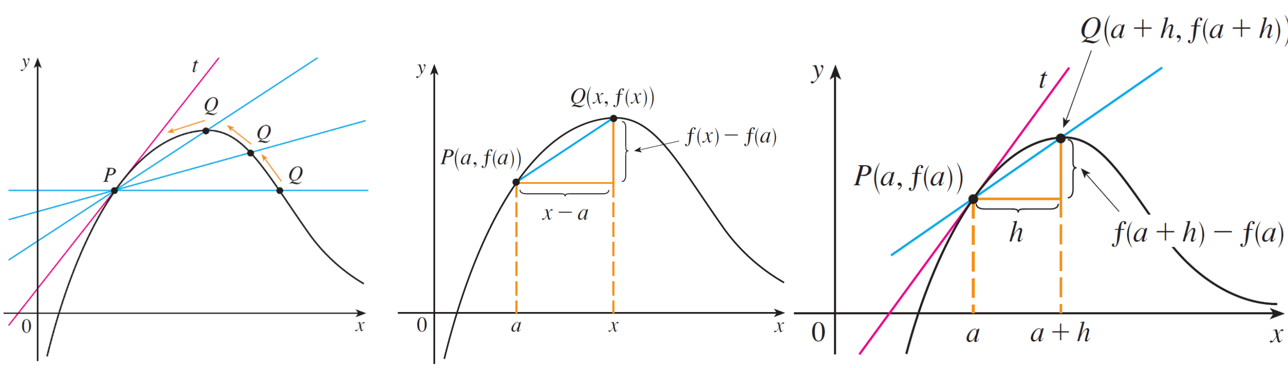
DEFINITION: The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \tag{1}$$

provided that this limit exists.

There is another (equivalent) expression for the slope of the tangent line:

$$\boxed{m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}} \tag{2}$$



EXAMPLE: Consider the graph of $y = x^2$.

(a) Find the slope of the tangent line to the graph at the point $P(1,1)$.

Solution 1: We have

$$
\begin{aligned}
m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{(a+h)^2-a^2}{h} &= \lim_{h\to 0}\frac{a^2+2ah+h^2-a^2}{h} \\
&= \lim_{h\to 0}\frac{2ah+h^2}{h} \\
&= \lim_{h\to 0}\frac{h(2a+h)}{h} \\
&= \lim_{h\to 0}(2a+h) \\
&= 2a+0 \\
&= 2a
\end{aligned}
$$

or

$$
\begin{aligned}
m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{(a+h)^2-a^2}{h} &= \lim_{h\to 0}\frac{(a+h-a)(a+h+a)}{h} \\
&= \lim_{h\to 0}\frac{h(2a+h)}{h} \\
&= \lim_{h\to 0}(2a+h) \\
&= 2a+0 \\
&= 2a
\end{aligned}
$$

So the slope of the tangent line at the point $P(1,1)$ is $m = 2\cdot 1 = 2$.

Solution 2: We have

$$
m = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x-a)(x+a)}{x-a} = \lim_{x\to a}(x+a) = a+a = 2a
$$

and the same result follows.

(b) Find the equation of the tangent line.

Solution: The equation of the tangent line can be found with the point-slope form of the equation of a line:

$$
\begin{aligned}
y - y_0 &= m(x-x_0) \\
y - 1 &= 2(x-1) \\
y - 1 &= 2x-2 \\
y &= 2x-2+1 = 2x-1
\end{aligned}
$$

EXAMPLE: Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Solution: We have

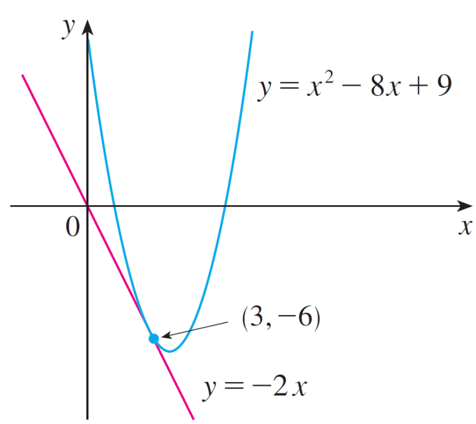
$$\begin{aligned} m &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{(x^2 - 8x + 9) - (a^2 - 8a + 9)}{x - a} \\ &= \lim_{x \to a} \frac{x^2 - 8x + 9 - a^2 + 8a - 9}{x - a} \\ &= \lim_{x \to a} \frac{x^2 - 8x - a^2 + 8a}{x - a} \\ &= \lim_{x \to a} \frac{x^2 - a^2 - 8x + 8a}{x - a} \\ &= \lim_{x \to a} \frac{(x - a)(x + a) - 8(x - a)}{x - a} \\ &= \lim_{x \to a} \frac{(x - a)[(x + a) - 8]}{x - a} \\ &= \lim_{x \to a} [(x + a) - 8] \\ &= 2a - 8 \end{aligned}$$

or

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a + h)^2 - 8(a + h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} \\ &= \lim_{h \to 0} \frac{h(2a + h - 8)}{h} \\ &= \lim_{h \to 0} (2a + h - 8) \\ &= 2a + 0 - 8 \\ &= 2a - 8 \end{aligned}$$

So the slope of the tangent line at the point $(3, -6)$ is $m = 2 \cdot 3 - 8 = -2$. The equation of the tangent line at this point can be found with the point-slope form of the equation of a line:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - (-6) &= -2(x - 3) \\ y + 6 &= -2x + 6 \\ y &= -2x \end{aligned}$$



Secant lines and tangent lines (or, more precisely, their slopes) are the geometric analogues of the average and instantaneous rates of change studied in the previous section, as summarized in the following chart:

Quantity	Algebraic Interpretation	Geometric Interpretation
$\dfrac{f(a+h)-f(a)}{h}$	Average rate of change of f from $x=a$ to $x=a+h$	Slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$
$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$	Instantaneous rate of change of f at $x=a$	Slope of the tangent line to the graph of f at $(a, f(a))$

Derivatives

We have seen that limits of the form

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad \text{or} \quad \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

arise in finding the slope of a tangent line or the velocity of an object. Moreover, the same type of limit arises whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

DEFINITION: The **derivative of a function *f* at a number *a***, denoted by $f'(a)$, is

$$\boxed{f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

if this limit exists.

REMARK: Equivalently,

$$\boxed{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

The derivative function f' has as its domain all the points at which the specified limit exists, and the value of the derivative function at the number x is the number $f'(x)$.

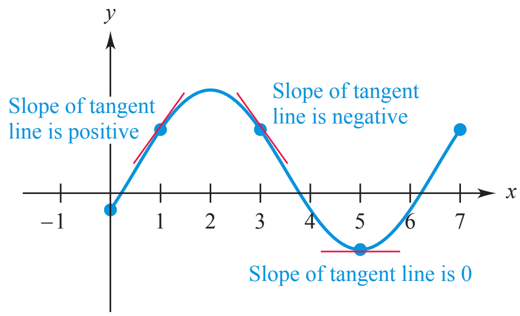
If $y=f(x)$ is a function, then its derivative is denoted either by f' or by y'. If x is a number in the domain of $y=f(x)$ such that $y'=f'(x)$ is defined, then the function f is said to be **differentiable** at x. The process that produces the function f' from the function f is called **differentiation**.

The derivative function may be interpreted in many ways, two of which were already discussed:

1. The derivative function f' gives the *instantaneous rate of change* of $y=f(x)$ with respect to x. This instantaneous rate of change can be interpreted as marginal cost, marginal revenue, or marginal profit (if the original function represents cost, revenue, or profit, respectively) or as velocity (if the original function represents displacement along a line). From now on, we will use "rate of change" to mean "instantaneous rate of change."

2. The derivative function f' gives the *slope* of the graph of f at any point. If the derivative is evaluated at $x=a$, then f' is the slope of the tangent line to the curve at the point $(a, f(a))$.

EXAMPLE: Use the graph of the function $f(x)$ in the Figure below to answer the given questions.



(a) Is $f'(3)$ positive or negative?

Solution: We know that $f'(3)$ is the slope of the tangent line to the graph at the point where $x = 3$. The Figure above shows that this tangent line slants downward from left to right, meaning that its slope is negative. Hence, $f'(3) < 0$.

(b) Which is larger, $f'(1)$ or $f'(5)$?

Solution: The Figure above shows that the tangent line to the graph at the point where $x = 1$ slants upward from left to right, meaning that its slope, $f'(1)$, is a positive number. The tangent line at the point where $x = 5$ is horizontal, so that it has slope 0. (That is, $f'(5) = 0$). Therefore, $f'(1) > f'(5)$.

(c) For what values of x is $f'(x)$ positive?

Solution: On the graph, find the points where the tangent line has positive slope (slants upward from left to right). At each such point, $f'(x) > 0$. The Figure above shows that this occurs when $0 < x < 2$ and when $5 < x < 7$.

EXAMPLE: If $f(x) = 3x - 5$, find $f'(x)$.

Solution 1: We have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\Big(3(a+h) - 5\Big) - (3a-5)}{h}$$

$$= \lim_{h\to 0} \frac{(3a + 3h - 5) - (3a - 5)}{h}$$

$$= \lim_{h\to 0} \frac{3a + 3h - 5 - 3a + 5}{h} = \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

Solution 2: We have

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{(3x-5) - (3a-5)}{x - a}$$

$$= \lim_{x\to a} \frac{3x - 5 - 3a + 5}{x - a} = \lim_{x\to a} \frac{3x - 3a}{x - a} = \lim_{x\to a} \frac{3(x-a)}{x-a} = \lim_{x\to a} 3 = 3$$

EXAMPLE: If $f(x)=\sqrt{x}$, find $f'(x)$.

Solution 1: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})}$$
$$=\lim_{x\to a}\frac{(\sqrt{x})^2-(\sqrt{a})^2}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}$$
$$=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

Solution $1'$: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x})^2-(\sqrt{a})^2}$$
$$=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

Solution 2: We have

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})}$$
$$=\lim_{h\to 0}\frac{(\sqrt{a+h})^2-(\sqrt{a})^2}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to 0}\frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to 0}\frac{h}{h(\sqrt{a+h}+\sqrt{a})}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{\sqrt{a+0}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

Solution $2'$: We have

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{a+h-a}$$
$$=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{(\sqrt{a+h})^2-(\sqrt{a})^2}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{\sqrt{a+0}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

EXAMPLE: If $f(x)=2x^3-7x$, find $f'(x)$.

Solution: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{(2x^3-7x)-(2a^3-7a)}{x-a}=\lim_{x\to a}\frac{2x^3-7x-2a^3+7a}{x-a}$$
$$=\lim_{x\to a}\frac{2x^3-2a^3-7x+7a}{x-a}=\lim_{x\to a}\frac{2(x^3-a^3)-7(x-a)}{x-a}$$
$$=\lim_{x\to a}\frac{2(x-a)(x^2+xa+a^2)-7(x-a)}{x-a}=\lim_{x\to a}\frac{(x-a)\Big(2(x^2+xa+a^2)-7\Big)}{x-a}$$
$$=\lim_{x\to a}\Big(2(x^2+xa+a^2)-7\Big)=2(a^2+a\cdot a+a^2)-7=2(3a^2)-7=6a^2-7$$

EXAMPLE: If $f(x)=\dfrac{3}{x}$, find $f'(x)$.

Solution 1: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\dfrac{3}{x}-\dfrac{3}{a}}{x-a}=\lim_{x\to a}\frac{xa\cdot\left(\dfrac{3}{x}-\dfrac{3}{a}\right)}{xa\cdot(x-a)}=\lim_{x\to a}\frac{xa\cdot\dfrac{3}{x}-xa\cdot\dfrac{3}{a}}{xa(x-a)}$$

$$=\lim_{x\to a}\frac{3a-3x}{xa(x-a)}=\lim_{x\to a}\frac{3(a-x)}{xa(x-a)}=\lim_{x\to a}\frac{-3(x-a)}{xa(x-a)}=\lim_{x\to a}\frac{-3}{xa}=-\frac{3}{a^2}$$

Solution 2: We have

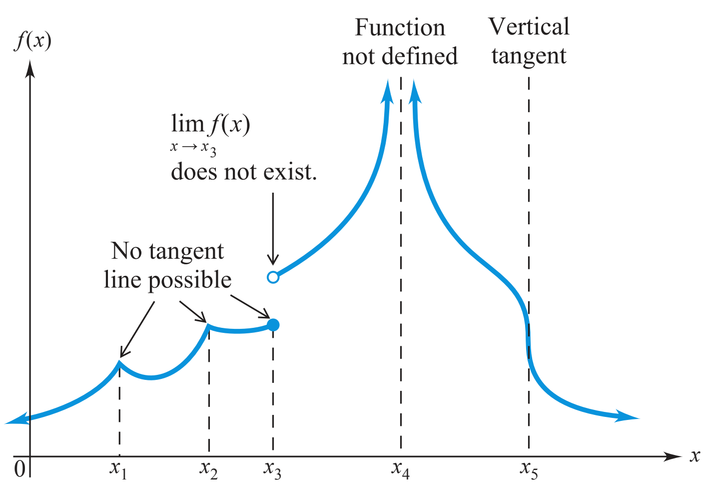
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\dfrac{3}{a+h}-\dfrac{3}{a}}{h}=\lim_{h\to 0}\frac{(a+h)a\cdot\left(\dfrac{3}{a+h}-\dfrac{3}{a}\right)}{(a+h)a\cdot h}$$

$$=\lim_{h\to 0}\frac{(a+h)a\cdot\dfrac{3}{a+h}-(a+h)a\cdot\dfrac{3}{a}}{(a+h)ah}=\lim_{h\to 0}\frac{3a-3(a+h)}{(a+h)ah}=\lim_{h\to 0}\frac{3a-3a-3h}{(a+h)ah}$$

$$=\lim_{h\to 0}\frac{-3h}{(a+h)ah}=\lim_{h\to 0}\frac{-3}{(a+h)a}=\frac{-3}{(a+0)a}=\frac{-3}{a\cdot a}=-\frac{3}{a^2}$$

Existence of the Derivative

The definition of the derivative includes the phrase "if this limit exists." If the limit used to define $f'(x)$ does not exist, then, of course, the derivative does not exist at that x. For example, a derivative cannot exist at a point where the function itself is not defined. If there is no function value for a particular value of x, there can be no tangent line for that value. Derivatives also do not exist at "corners" or "sharp points" on a graph. Since a vertical line has an undefined slope, the derivative cannot exist at any point where the tangent line is vertical. This figure summarizes various ways that a derivative can fail to exist.



EXAMPLE: Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Solution: Note that

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Therefore on the one hand we have

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

On the other hand,

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

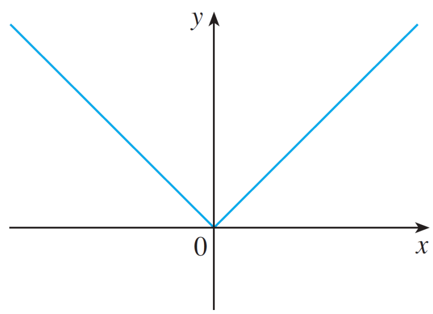
Since

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} \neq \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}$$

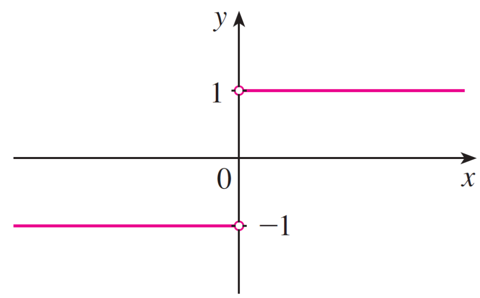
it follows that

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \quad \text{does not exist}$$

Therefore $f'(0)$ does not exist, so $f(x) = |x|$ is not differentiable at $x = 0$.



(a) $y = f(x) = |x|$



(b) $y = f'(x)$

Price Elasticity of Demand

Any retailer who sells a product or a service is concerned with how a change in price affects demand for the article. The sensitivity of demand to price changes varies with different items. For smaller items, such as soft drinks, food staples, and lightbulbs, small percentage changes in price will not affect the demand for the item much. However, sometimes a small percentage change in price on big-ticket items, such as cars, homes, and furniture, can have significant effects on demand.

One way to measure the sensitivity of changes in price to demand is by the ratio of percent change in demand to percent change in price. If q represents the quantity demanded and p the price of the item, this ratio can be written as

$$\frac{\Delta q/q}{\Delta p/p}$$

where Δq represents the change in q and Δp represents the change in p. The ratio is usually negative, because q and p are positive, while Δq and Δp typically have opposite signs. (An *increase* in price causes a *decrease* in demand.) If the absolute value of this quantity is large, it shows that a small increase in price can cause a relatively large decrease in demand.

Applying some algebra, we can rewrite this ratio as

$$\frac{\Delta q/q}{\Delta p/p} = \frac{(\Delta q/q)\cdot p\cdot q}{(\Delta p/p)\cdot p\cdot q} = \frac{(\Delta q)\cdot p}{(\Delta p)\cdot q} = \frac{\Delta q}{\Delta p}\cdot\frac{p}{q} = \frac{p}{q}\cdot\frac{\Delta q}{\Delta p}$$

or

$$\frac{\Delta q/q}{\Delta p/p} = \frac{\Delta q}{q}\div\frac{\Delta p}{p} = \frac{\Delta q}{q}\cdot\frac{p}{\Delta p} = \frac{\Delta q\cdot p}{q\cdot\Delta p} = \frac{p\cdot\Delta q}{q\cdot\Delta p} = \frac{p}{q}\cdot\frac{\Delta q}{\Delta p}$$

Suppose $q = f(p)$. (Note that this is the inverse of the way our demand functions have been expressed so far; previously, we had $p = D(q)$.) Then $\Delta q = f(p+\Delta p) - f(p)$. It follows that

$$\frac{\Delta q}{\Delta p} = \frac{f(p+\Delta p)-f(p)}{\Delta p}$$

As $\Delta p \to 0$, this quotient becomes

$$\lim_{\Delta p\to 0}\frac{\Delta q}{\Delta p} = \lim_{\Delta p\to 0}\frac{f(p+\Delta p)-f(p)}{\Delta p} = \frac{dq}{dp}$$

and

$$\lim_{\Delta p\to 0}\frac{p}{q}\cdot\frac{\Delta q}{\Delta p} = \frac{p}{q}\cdot\lim_{\Delta p\to 0}\frac{\Delta q}{\Delta p} = \frac{p}{q}\cdot\frac{dq}{dp}$$

The quantity

$$\boxed{E = -\frac{p}{q}\cdot\frac{dq}{dp}}$$

is positive because dq/dp is negative. E is called **elasticity of demand** and measures the instantaneous responsiveness of demand to price. For example, E may be .6 for physician services (expenses that have considerable price increases each year, but still have a high demand), but may be 2.3 for restaurant meals (highly enjoyable, but high-cost items and not necessities). These numbers indicate that the demand for physician services is much less responsive to price changes than the demand for restaurant meals. Another factor that impacts elasticity of demand is the availability of substitute products. For example, if one airline increases prices on a particular route but other airlines serving the route do not, then rather than paying the higher prices, consumers could fly with a different airline (the substitute product).

If $E < 1$, the relative change in demand is less than the relative change in price, and the demand is called **inelastic**. If $E > 1$, the relative change in demand is greater than the relative change in price, and the demand is called **elastic**. When $E = 1$, the percentage changes in price and demand are relatively equal, and the demand is said to have **unit elasticity**.

Elasticity of Demand

Let $q = f(p)$, where q is the demand at a price p. The **elasticity of demand** is as follows:

Demand is **inelastic** if $E < 1$.

Demand is **elastic** if $E > 1$.

Demand has **unit elasticity** if $E = 1$.

Sometimes elasticity is counterintuitive. The addiction to illicit drugs is an excellent example. The quantity of the drug demanded by addicts, if anything, increases, no matter what the cost. Thus, illegal drugs are an inelastic commodity.

EXAMPLE: Suppose that the demand for flat screen televisions is expressed by the equation

$$q = -.025p + 20.45$$

where q is the annual demand (in millions of televisions) and p is the price of the product (in dollars).

(a) Calculate and interpret the elasticity of demand when $p = \$200$ and when $p = \$500$.

Solution: Since $q = -.025p + 20.45$, we have $dq/dp = -.025$, so that

$$E = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p}{-.025p + 20.45} \cdot (-.025) = \frac{.025p}{-.025p + 20.45}$$

Let $p = 200$ to get

$$E = \frac{.025(200)}{-.025(200) + 20.45} \approx .324$$

Since $.324 < 1$, the demand was inelastic, and a percentage change in price resulted in a smaller percentage change in demand. For example, a 10% increase in price will cause a 3.24% decrease in demand.

If $p = 500$, then

$$E = \frac{.025(500)}{-.025(500) + 20.45} \approx 1.57$$

Since $1.57 > 1$, the price is elastic. At this point, a percentage increase in price resulted in a greater percentage decrease in demand. A 10% increase in price resulted in a 15.7% decrease in demand.

(b) Determine the price at which demand had unit elasticity ($E = 1$). What is the significance of this price?

Solution 1: Demand had unit elasticity at the price p that made $E = 1$, so we must solve the equation

$$E = \frac{.025p}{-.025p + 20.45} = 1$$

$$.025p = -.025p + 20.45$$

$$.05p = 20.45$$

$$p = \frac{20.45}{.05} = 409$$

Demand had unit elasticity at a price of \$409 per flat screen television. Unit elasticity indicates that the changes in price and demand are about the same.

Solution 2: Recall that

$$\text{Revenue} = (\text{Price per item}) \times (\text{Number of items})$$

We have

$$\text{Revenue } = p \cdot q = p(-.025p + 20.45) = -.025p^2 + 20.45p$$

The quadratic function $R = -.025p^2 + 20.45p$ attains its maximum value if

$$p = -\frac{b}{2a} = -\frac{20.45}{2(-.025)} = -\frac{20.45}{-.05} = 409$$

which gives us the same result follows.

REMARK: Revenue is maximized when price is set so that elasticity of demand is exactly one.

Appendix

EXAMPLE: If $f(x) = 3 - 2x - 7x^2$, find $f'(x)$.

Solution 1: We have

$$\begin{aligned}
f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{\Big(3-2(a+h)-7(a+h)^2\Big)-\Big(3-2a-7a^2\Big)}{h}\\
&= \lim_{h\to 0}\frac{3-2a-2h-7(a^2+2ah+h^2)-3+2a+7a^2}{h}\\
&= \lim_{h\to 0}\frac{3-2a-2h-7a^2-14ah-7h^2-3+2a+7a^2}{h}\\
&= \lim_{h\to 0}\frac{-2h-14ah-7h^2}{h}\\
&= \lim_{h\to 0}\frac{h(-2-14a-7h)}{h}\\
&= \lim_{h\to 0}(-2-14a-7h)\\
&= -2-14a-7\cdot 0\\
&= -2-14a
\end{aligned}$$

Solution 2: We have

$$\begin{aligned}
f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{(3-2x-7x^2)-(3-2a-7a^2)}{x-a}\\
&= \lim_{x\to a}\frac{3-2x-7x^2-3+2a+7a^2}{x-a}\\
&= \lim_{x\to a}\frac{-2x-7x^2+2a+7a^2}{x-a}\\
&= \lim_{x\to a}\frac{-2x+2a-7x^2+7a^2}{x-a}\\
&= \lim_{x\to a}\frac{-2(x-a)-7(x^2-a^2)}{x-a}\\
&= \lim_{x\to a}\frac{-2(x-a)-7(x-a)(x+a)}{x-a}\\
&= \lim_{x\to a}\frac{(x-a)\Big(-2-7(x+a)\Big)}{x-a}\\
&= \lim_{x\to a}\Big(-2-7(x+a)\Big)\\
&= -2-7(a+a) = -2-7(2a) = -2-14a
\end{aligned}$$

EXAMPLE: If $f(x)=\sqrt{5x+9}$, find $f'(x)$.

Solution 1: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sqrt{5x+9}-\sqrt{5a+9}}{x-a}$$

$$=\lim_{x\to a}\frac{(\sqrt{5x+9}-\sqrt{5a+9})(\sqrt{5x+9}+\sqrt{5a+9})}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$=\lim_{x\to a}\frac{(\sqrt{5x+9})^2-(\sqrt{5a+9})^2}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$=\lim_{x\to a}\frac{(5x+9)-(5a+9)}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$=\lim_{x\to a}\frac{5x+9-5a-9}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$=\lim_{x\to a}\frac{5x-5a}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$=\lim_{x\to a}\frac{5(x-a)}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$=\lim_{x\to a}\frac{5}{\sqrt{5x+9}+\sqrt{5a+9}}$$

$$=\frac{5}{\sqrt{5a+9}+\sqrt{5a+9}}$$

$$=\frac{5}{2\sqrt{5a+9}}$$

In short,

$$f'(a)=\lim_{x\to a}\frac{\sqrt{5x+9}-\sqrt{5a+9}}{x-a}=\lim_{x\to a}\frac{(\sqrt{5x+9}-\sqrt{5a+9})(\sqrt{5x+9}+\sqrt{5a+9})}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$=\lim_{x\to a}\frac{(\sqrt{5x+9})^2-(\sqrt{5a+9})^2}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}=\lim_{x\to a}\frac{(5x+9)-(5a+9)}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$=\lim_{x\to a}\frac{5(x-a)}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}=\lim_{x\to a}\frac{5}{\sqrt{5x+9}+\sqrt{5a+9}}=\frac{5}{2\sqrt{5a+9}}$$

Solution 2: We have

$$
\begin{aligned}
f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{\sqrt{5(a+h)+9}-\sqrt{5a+9}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{5(a+h)+9}-\sqrt{5a+9})(\sqrt{5(a+h)+9}+\sqrt{5a+9})}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} \\
&= \lim_{h\to 0} \frac{(\sqrt{5(a+h)+9})^2-(\sqrt{5a+9})^2}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} \\
&= \lim_{h\to 0} \frac{(5(a+h)+9)-(5a+9)}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} \\
&= \lim_{h\to 0} \frac{5a+5h+9-5a-9}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} \\
&= \lim_{h\to 0} \frac{5h}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} \\
&= \lim_{h\to 0} \frac{5}{\sqrt{5(a+h)+9}+\sqrt{5a+9}} \\
&= \frac{5}{\sqrt{5(a+0)+9}+\sqrt{5a+9}} \\
&= \frac{5}{\sqrt{5a+9}+\sqrt{5a+9}} \\
&= \frac{5}{2\sqrt{5a+9}}
\end{aligned}
$$

In short,

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sqrt{5(a+h)+9}-\sqrt{5a+9}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{5(a+h)+9}-\sqrt{5a+9})(\sqrt{5(a+h)+9}+\sqrt{5a+9})}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} \\
&= \lim_{h\to 0} \frac{(\sqrt{5(a+h)+9})^2-(\sqrt{5a+9})^2}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} = \lim_{h\to 0} \frac{(5(a+h)+9)-(5a+9)}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} \\
&= \lim_{h\to 0} \frac{5h}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} = \lim_{h\to 0} \frac{5}{\sqrt{5(a+h)+9}+\sqrt{5a+9}} = \frac{5}{2\sqrt{5a+9}}
\end{aligned}
$$

EXAMPLE: If $f(x)=\sqrt{2-3x}$, find $f'(x)$.

Solution: We have

$$\begin{aligned}
f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}&=\lim_{x\to a}\frac{\sqrt{2-3x}-\sqrt{2-3a}}{x-a}\\
&=\lim_{x\to a}\frac{(\sqrt{2-3x}-\sqrt{2-3a})(\sqrt{2-3x}+\sqrt{2-3a})}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{(\sqrt{2-3x})^2-(\sqrt{2-3a})^2}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{(2-3x)-(2-3a)}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{2-3x-2+3a}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{-3x+3a}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{-3(x-a)}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{-3}{\sqrt{2-3x}+\sqrt{2-3a}}\\
&=\frac{-3}{\sqrt{2-3a}+\sqrt{2-3a}}\\
&=-\frac{3}{2\sqrt{2-3a}}
\end{aligned}$$

In short,

$$f'(a)=\lim_{x\to a}\frac{\sqrt{2-3x}-\sqrt{2-3a}}{x-a}=\lim_{x\to a}\frac{(\sqrt{2-3x}-\sqrt{2-3a})(\sqrt{2-3x}+\sqrt{2-3a})}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}$$

$$=\lim_{x\to a}\frac{(\sqrt{2-3x})^2-(\sqrt{2-3a})^2}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}=\lim_{x\to a}\frac{(2-3x)-(2-3a)}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}$$

$$=\lim_{x\to a}\frac{-3(x-a)}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}=\lim_{x\to a}\frac{-3}{\sqrt{2-3x}+\sqrt{2-3a}}=-\frac{3}{2\sqrt{2-3a}}$$

EXAMPLE: If $f(x) = \dfrac{x+3}{4-x}$, find $f'(x)$.

Solution 1: We have

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\dfrac{x+3}{4-x} - \dfrac{a+3}{4-a}}{x-a} = \lim_{x\to a} \frac{\left(\dfrac{x+3}{4-x} - \dfrac{a+3}{4-a}\right)\cdot(4-x)(4-a)}{(x-a)\cdot(4-x)(4-a)}$$

$$= \lim_{x\to a} \frac{\dfrac{x+3}{4-x}\cdot(4-x)(4-a) - \dfrac{a+3}{4-a}\cdot(4-x)(4-a)}{(x-a)\cdot(4-x)(4-a)}$$

$$= \lim_{x\to a} \frac{(x+3)(4-a)-(a+3)(4-x)}{(x-a)(4-x)(4-a)} = \lim_{x\to a} \frac{(4x-xa+12-3a)-(4a-ax+12-3x)}{(x-a)(4-x)(4-a)}$$

$$= \lim_{x\to a} \frac{4x-xa+12-3a-4a+ax-12+3x}{(x-a)(4-x)(4-a)} = \lim_{x\to a} \frac{7x-7a}{(x-a)(4-x)(4-a)}$$

$$= \lim_{x\to a} \frac{7(x-a)}{(x-a)(4-x)(4-a)} = \lim_{x\to a} \frac{7}{(4-x)(4-a)} = \frac{7}{(4-a)(4-a)} = \frac{7}{(4-a)^2}$$

Solution 2: We have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{(a+h)+3}{4-(a+h)} - \dfrac{a+3}{4-a}}{h} = \lim_{h\to 0} \frac{\dfrac{a+h+3}{4-a-h} - \dfrac{a+3}{4-a}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{a+h+3}{4-a-h} - \dfrac{a+3}{4-a}\right)\cdot(4-a-h)(4-a)}{h\cdot(4-a-h)(4-a)}$$

$$= \lim_{h\to 0} \frac{\dfrac{a+h+3}{4-a-h}\cdot(4-a-h)(4-a) - \dfrac{a+3}{4-a}\cdot(4-a-h)(4-a)}{h\cdot(4-a-h)(4-a)}$$

$$= \lim_{h\to 0} \frac{(a+h+3)(4-a)-(a+3)(4-a-h)}{h(4-a-h)(4-a)}$$

$$= \lim_{h\to 0} \frac{(4a-a^2+4h-ah+12-3a)-(4a-a^2-ah+12-3a-3h)}{h(4-a-h)(4-a)}$$

$$= \lim_{h\to 0} \frac{7h}{h(4-a-h)(4-a)} = \lim_{h\to 0} \frac{7}{(4-a-h)(4-a)} = \frac{7}{(4-a-0)(4-a)} = \frac{7}{(4-a)^2}$$