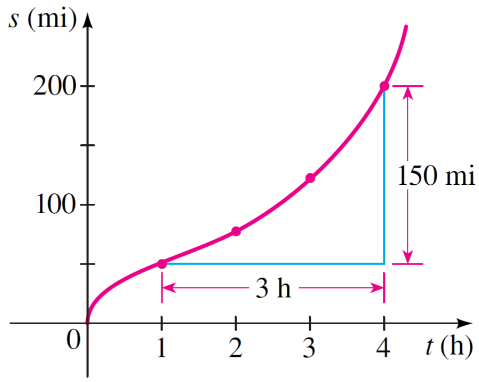
Section 11.3 Rates of Change

Suppose you take a car trip and record the distance that you travel every few minutes. The distance s you have traveled is a function of the time t:

$$s(t) = \text{ total distance traveled at time } t$$

We graph the function s as shown in the Figure below.



The graph shows that you have traveled a total of 50 miles after 1 hour, 75 miles after 2 hours, 140 miles after 3 hours, and so on. To find your *average* speed between any two points on the trip, we divide the distance traveled by the time elapsed.

Let's calculate your average speed between 1:00 P.M. and 4:00 P.M. The time elapsed is $4-1 = 3$ hours. To find the distance you traveled, we subtract the distance at 1:00 P.M. from the distance at 4:00 P.M., that is, $200 - 50 = 150$ mi. Thus, your average speed is

$$\text{average speed } = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{150 \text{ mi}}{3 \text{ h}} = 50 \text{ mi/h}$$

The average speed we have just calculated can be expressed using function notation:

$$\text{average speed } = \frac{s(4) - s(1)}{4 - 1} = \frac{200 - 50}{3} = 50 \text{ mi/h}$$

Note that the average speed is different over different time intervals. For example, between 2:00 P.M. and 3:00 P.M. we find that

$$\text{average speed } = \frac{s(3) - s(2)}{3 - 2} = \frac{140 - 75}{1} = 65 \text{ mi/h}$$

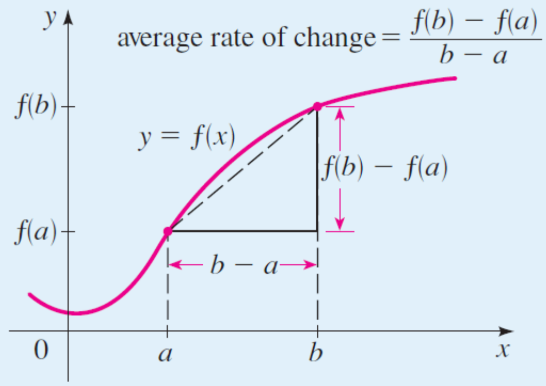
Finding average rates of change is important in many contexts. For instance, we may be interested in knowing how quickly the air temperature is dropping as a storm approaches, or how fast revenues are increasing from the sale of a new product. So we need to know how to determine the average rate of change of the functions that model these quantities. In fact, the concept of average rate of change can be defined for any function.

Average Rate of Change

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

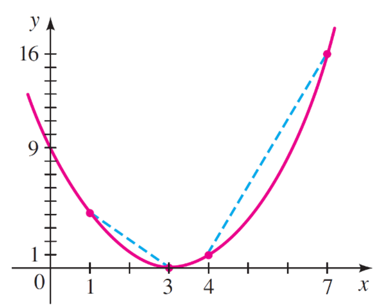
The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of f, that is, the line that passes through $(a, f(a))$ and $(b, f(b))$.



EXAMPLE: For the function $f(x) = (x - 3)^2$, whose graph is shown in the Figure below, find the average rate of change between the following points:

(a) $x = 1$ and $x = 3$

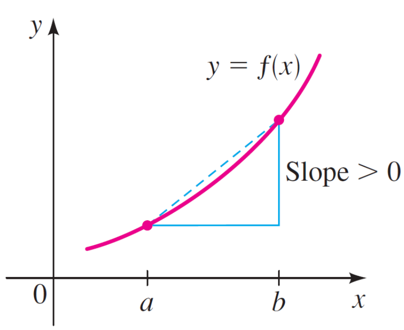
(b) $x = 4$ and $x = 7$



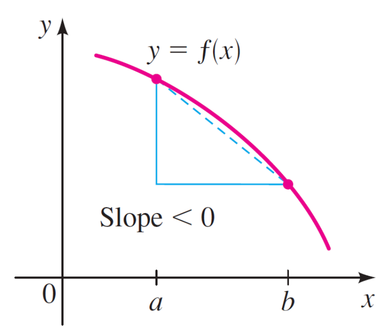
Solution:

(a) Average rate of change $= \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{(3-3)^2 - (1-3)^2}{3 - 1} = \dfrac{0 - 4}{2} = -2$

(b) Average rate of change $= \dfrac{f(7) - f(4)}{7 - 4} = \dfrac{(7-3)^2 - (4-3)^2}{7 - 4} = \dfrac{16 - 1}{3} = 5$



f increasing
Average rate of change positive



f decreasing
Average rate of change negative

EXAMPLE: If an object is dropped from a tall building, then the distance it has fallen after t seconds is given by the function $d(t) = 16t^2$. Find its average speed (average rate of change) over the following intervals:

(a) Between 1 s and 5 s (b) Between $t = a$ and $t = a + h$

Solution:

(a) Average rate of change $= \dfrac{d(5) - d(1)}{5 - 1} = \dfrac{16(5)^2 - 16(1)^2}{5 - 1} = \dfrac{400 - 16}{4} = 96$ ft/s

(b) Average rate of change
$$= \frac{d(a+h) - d(a)}{(a+h) - a} = \frac{16(a+h)^2 - 16(a)^2}{(a+h) - a} = \frac{16(a^2 + 2ah + h^2 - a^2)}{h}$$
$$= \frac{16(2ah + h^2)}{h}$$
$$= \frac{16h(2a + h)}{h}$$
$$= 16(2a + h)$$

EXAMPLE: Let $f(x) = 3x - 5$. Find the average rate of change of f between the following points.

(a) $x = 0$ and $x = 1$ (b) $x = 3$ and $x = 7$ (c) $x = a$ and $x = a + h$

What conclusion can you draw from your answers?

Solution:

(a) Average rate of change $= \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{(3 \cdot 1 - 5) - (3 \cdot 0 - 5)}{1} = \dfrac{(-2) - (-5)}{1} = 3$

(b) Average rate of change $= \dfrac{f(7) - f(3)}{7 - 3} = \dfrac{(3 \cdot 7 - 5) - (3 \cdot 3 - 5)}{4} = \dfrac{16 - 4}{4} = 3$

(c) Average rate of change
$$= \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{[3(a+h) - 5] - [3a - 5]}{h} = \frac{3a + 3h - 5 - 3a + 5}{h}$$
$$= \frac{3h}{h} = 3$$

It appears that the average rate of change is always 3 for this function. In fact, part (c) proves that the rate of change between any two arbitrary points $x = a$ and $x = a + h$ is 3.

EXAMPLE: Let $f(x) = mx + b$. Find the average rate of change of f between the points $x = a$ and $x = a + h$.

Solution: We have

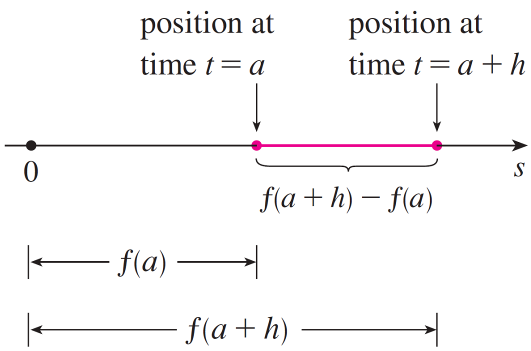
$$\text{Average rate of change} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{[m(a+h) + b] - [ma + b]}{h}$$
$$= \frac{ma + mh + b - ma - b}{h} = \frac{mh}{h} = m$$

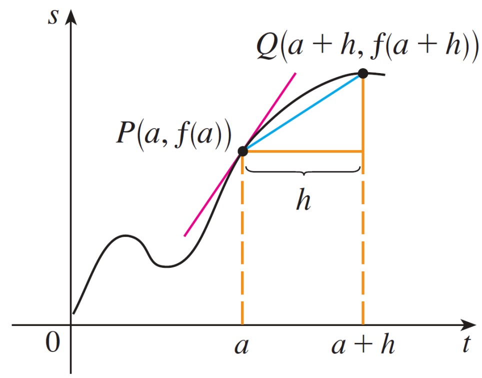
Instantaneous Rate of Change

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a+h) - f(a)$. The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ in the second figure.





Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a+h]$. In other words, we let h approach 0. We define **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

$$\boxed{v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

EXAMPLE: Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground. What is the velocity of the ball after 5 seconds?

Solution: We use the equation of motion $s = f(t) = 4.9t^2$, where t is time (in seconds) and s is the displacement (in meters) to find the velocity $v(a)$ after a seconds:

$$\begin{aligned} v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h} &= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2) - 4.9a^2}{h} \\ &= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} \\ &= \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h} \\ &= \lim_{h \to 0} \frac{4.9h(2a + h)}{h} \\ &= \lim_{h \to 0} 4.9(2a + h) \\ &= 4.9(2a + 0) = 4.9(2a) = 9.8a \end{aligned}$$

Therefore the velocity after 5 seconds is $v(5) = 9.8 \cdot 5 = 49$ m/s.

EXAMPLE: A company determines that the cost (in hundreds of dollars) of manufacturing x cases of computer mice is

$$C(x) = -.2x^2 + 8x + 40 \quad (0 < x < 20)$$

(a) Find the average rate of change of cost for manufacturing between 5 and 10 cases.

Solution: Use the formula for average rate of change. The cost to manufacture 5 cases is

$$C(5) = -.2(5^2) + 8(5) + 40 = 75$$

or \$7500. The cost to manufacture 10 cases is

$$C(10) = -.2(102) + 8(10) + 40 = 100$$

or \$10,000. The average rate of change of cost is

$$\frac{C(10) - C(5)}{10 - 5} = \frac{100 - 75}{5} = 5$$

Thus, on the average, cost is increasing at the rate of \$500 per case when production is increased from 5 to 10 cases.

(b) Find the instantaneous rate of change with respect to the number of cases produced when 5 cases are produced.

Solution: We first note that since $C(x) = -.2x^2 + 8x + 40$, it follows that

$$C(5+h) = -.2(5+h)^2 + 8(5+h) + 40 \quad \text{and} \quad C(5) = -.2(5^2) + 8(5) + 40$$

Therefore the instantaneous rate of change when $x = 5$ is given by

$$\begin{aligned}
\lim_{h\to 0} \frac{C(5+h) - C(5)}{h} &= \lim_{h\to 0} \frac{[-.2(5+h)^2 + 8(5+h) + 40] - [-.2(5^2) + 8(5) + 40]}{h} \\
&= \lim_{h\to 0} \frac{[-.2(5^2 + 2\cdot 5\cdot h + h^2) + 8(5+h) + 40] - [-.2(5^2) + 8(5) + 40]}{h} \\
&= \lim_{h\to 0} \frac{-.2(5^2) - .2\cdot 2\cdot 5\cdot h - .2\cdot h^2 + 8(5) + 8h + 40 + .2(5^2) - 8(5) - 40}{h} \\
&= \lim_{h\to 0} \frac{-.2\cdot 2\cdot 5\cdot h - .2\cdot h^2 + 8h}{h} \\
&= \lim_{h\to 0}(-.2\cdot 2\cdot 5 - .2\cdot h + 8) \\
&= -.2\cdot 2\cdot 5 - .2\cdot 0 + 8 \\
&= -2 - 0 + 8 \\
&= 6
\end{aligned}$$

When 5 cases are manufactured, the cost is increasing at the rate of \$600 per case.

REMARK: The rate of change of the cost function is called the **marginal cost**. Similarly, **marginal revenue** and **marginal profit** are the rates of change of the revenue and profit functions, respectively. Part (b) of the Example above shows that the marginal cost when 5 cases are manufactured is \$600 per case.