Section 11.2 One-sided Limits and Limits Involving Infinity

One-sided Limits

DEFINITION: We write

$$\boxed{\lim_{x\to a^-} f(x) = L}$$

and say the **left-hand limit of $f(x)$ as x approaches a** is equal to L if we can make the values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and x less than a.

Similarly, we write

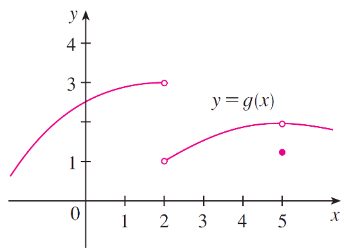
$$\boxed{\lim_{x\to a^+} f(x) = L}$$

and say the **right-hand limit of $f(x)$ as x approaches a** is equal to L if we can make the values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and x greater than a.

IMPORTANT:

$$\boxed{\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L}$$

EXAMPLE: The graph of a function g is shown below.



Then

(a) $\lim_{x\to 2^-} g(x) = 3$ (b) $\lim_{x\to 2^+} g(x) = 1$ (c) $\lim_{x\to 2} g(x)$ D.N.E.

(d) $\lim_{x\to 5^-} g(x) = 2$ (e) $\lim_{x\to 5^+} g(x) = 2$ (f) $\lim_{x\to 5} g(x) = 2$

EXAMPLE: Let

$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ x+1 & \text{if } x > 2 \end{cases}$$

Then

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 \overset{DSP}{=} 2^2 = 4 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+1) \overset{DSP}{=} 2+1 = 3.$$

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, it follows that $\lim_{x\to 2} f(x)$ does not exist.

EXAMPLE: Let

$$g(x) = \begin{cases} \dfrac{x+2}{x-1} & \text{if } x \le -1 \\ x^3 & \text{if } -1 < x \le 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

Then

(a) $\lim_{x\to -1^-} g(x) = \dfrac{-1+2}{-1-1} = -\dfrac{1}{2}$ (b) $\lim_{x\to -1^+} g(x) = (-1)^3 = -1$ (c) $\lim_{x\to -1} g(x)$ D.N.E. (d) $g(-1) = -\dfrac{1}{2}$

(e) $\lim_{x\to 0^-} g(x) = 0^3 = 0$ (f) $\lim_{x\to 0^+} g(x) = \sqrt{0} = 0$ (g) $\lim_{x\to 0} g(x) = 0$ (h) $g(0) = 0^3 = 0$

EXAMPLE: Find each of the given limits.

(a) $\lim\limits_{x\to 2^+}\sqrt{4-x^2}$ and $\lim\limits_{x\to 2^-}\sqrt{4-x^2}$

Solution: Since $f(x)=\sqrt{4-x^2}$ is not defined when $x>2$, the right-hand limit (which requires that $x>2$) does not exist. For the left-hand limit, write the square root in exponential form and apply the appropriate limit properties

$$\lim_{x\to 2^-}\sqrt{4-x^2}=\lim_{x\to 2^-}(4-x^2)^{1/2}=[\lim_{x\to 2^-}(4-x^2)]^{1/2}=(4-2^2)^{1/2}=0^{1/2}=0$$

(b) $\lim\limits_{x\to 3^+}[\sqrt{x-3}+x^2+1]$

Solution: We have

$$\begin{aligned}\lim_{x\to 3^+}[\sqrt{x-3}+x^2+1] &= \lim_{x\to 3^+}[(x-3)^{1/2}+x^2+1]\\ &= \lim_{x\to 3^+}(x-3)^{1/2}+\lim_{x\to 3^+}x^2+\lim_{x\to 3^+}1\\ &= [\lim_{x\to 3^+}(x-3)]^{1/2}+\lim_{x\to 3^+}x^2+\lim_{x\to 3^+}1=(3-3)^{1/2}+3^2+1=0+9+1=10\end{aligned}$$

Infinite Limits

DEFINITION: The notation

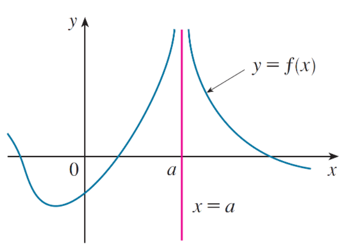
$$\boxed{\lim_{x\to a}f(x)=\infty}$$

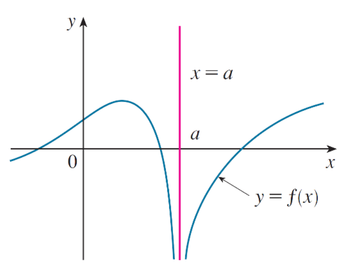
means that the values of $f(x)$ can be made arbitrary large (as large as we like) by taking x sufficiently close to a (on either side of a) but not equal to a.

Similarly,

$$\boxed{\lim_{x\to a}f(x)=-\infty}$$

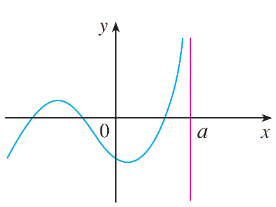
means that the values of $f(x)$ can be made as large negative as we like by taking x sufficiently close to a (on either side of a) but not equal to a.



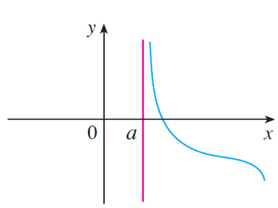


Similar definitions can be given for the one-sided infinite limits

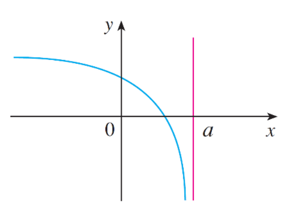
$$\boxed{\begin{array}{ll}\lim\limits_{x\to a^-}f(x)=\infty & \lim\limits_{x\to a^+}f(x)=\infty\\ \lim\limits_{x\to a^-}f(x)=-\infty & \lim\limits_{x\to a^+}f(x)=-\infty\end{array}}$$



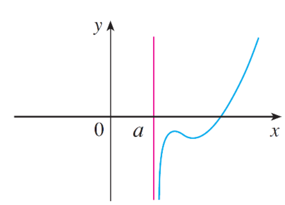
(a) $\lim\limits_{x\to a^-}f(x)=\infty$



(b) $\lim\limits_{x\to a^+}f(x)=\infty$



(c) $\lim\limits_{x\to a^-}f(x)=-\infty$



(d) $\lim\limits_{x\to a^+}f(x)=-\infty$

DEFINITION: The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

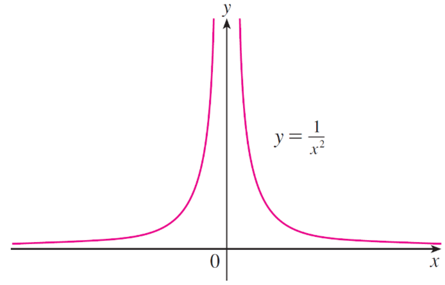
$\lim_{x \to a} f(x) = \infty$	$\lim_{x \to a^-} f(x) = \infty$	$\lim_{x \to a^+} f(x) = \infty$
$\lim_{x \to a} f(x) = -\infty$	$\lim_{x \to a^-} f(x) = -\infty$	$\lim_{x \to a^+} f(x) = -\infty$

EXAMPLES:

1. The limits $\lim_{x \to 0} \frac{1}{x^2}$, $\lim_{x \to 0^-} \frac{1}{x^2}$, $\lim_{x \to 0^+} \frac{1}{x^2}$ D.N.E., moreover

$$\lim_{x \to 0} \frac{1}{x^2} = \lim_{x \to 0^-} \frac{1}{x^2} = \lim_{x \to 0^+} \frac{1}{x^2} = \infty$$

x	$f(x)$
$\pm$ 0.1	100
$\pm$ 0.01	10000
$\pm$ 0.001	1000000
$\pm$ 0.0001	100000000
$\pm$ 0.00001	10000000000
$\pm$ 0.000001	1000000000000



It follows that $x = 0$ is the vertical asymptote of the function $f(x) = \frac{1}{x^2}$.

2. The limits $\lim_{x \to 5^-} \frac{1}{5 - x}$ and $\lim_{x \to 5^+} \frac{1}{5 - x}$ D.N.E., moreover

$$\lim_{x \to 5^-} \frac{1}{5 - x} = \left[\begin{array}{l} \text{WORK:} \\ \frac{1}{5 - 4.99} = \frac{1}{0.01} = \frac{+\text{“NOT SMALL”}}{+\text{“SMALL”}} = +\text{“BIG”} \end{array} \right] = \infty$$

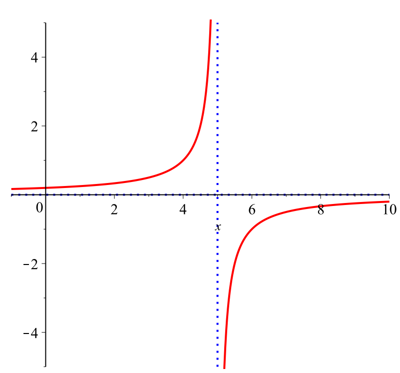
and

$$\lim_{x \to 5^+} \frac{1}{5 - x} = \left[\begin{array}{l} \text{WORK:} \\ \frac{1}{5 - 5.01} = \frac{1}{-0.01} = \frac{+\text{“NOT SMALL”}}{-\text{“SMALL”}} = -\text{“BIG”} \end{array} \right] = -\infty$$

It follows that

(a) $x = 5$ is the vertical asymptote of the function $f(x) = \frac{1}{5 - x}$;

(b) $\lim_{x \to 5} \frac{1}{5 - x}$ D.N.E. and neither ∞ nor $-\infty$.

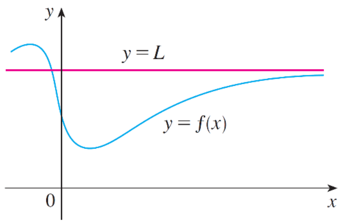


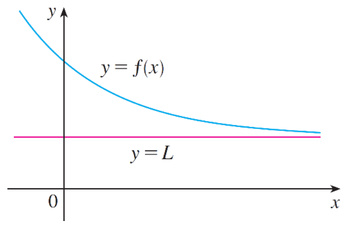
II. Limits at Infinity

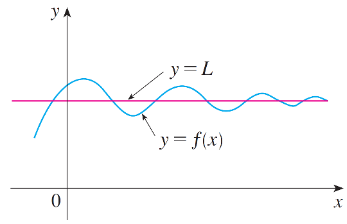
DEFINITION: Let f be a function defined on some interval (a, ∞). Then

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

means that the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.



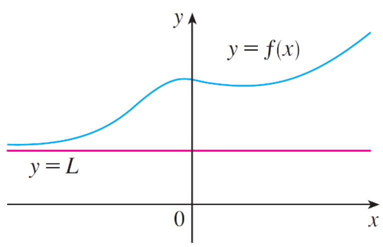


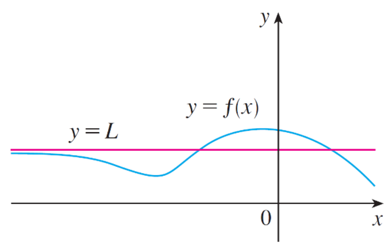


Similarly, the notation

$$\boxed{\lim_{x\to-\infty} f(x) = L}$$

means that the values of $f(x)$ can be made arbitrary close to L by taking x sufficiently large negative.





DEFINITION: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

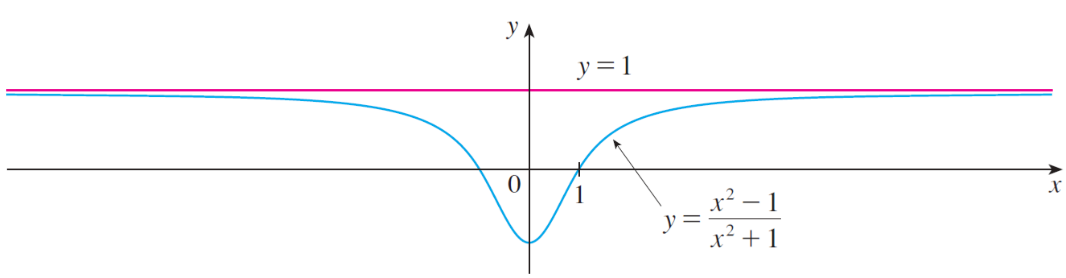
$$\boxed{\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L}$$

EXAMPLE: Find the horizontal asymptote of $f(x) = \dfrac{x^2-1}{x^2+1}$.

Solution: We have

$$\lim_{x\to\pm\infty}\frac{x^2-1}{x^2+1} = \left[\frac{\infty}{\infty}\right] \overset{A}{=} \lim_{x\to\pm\infty}\frac{\dfrac{x^2-1}{x^2}}{\dfrac{x^2+1}{x^2}} \overset{A}{=} \lim_{x\to\pm\infty}\frac{\dfrac{x^2}{x^2}-\dfrac{1}{x^2}}{\dfrac{x^2}{x^2}+\dfrac{1}{x^2}} \overset{A}{=} \lim_{x\to\pm\infty}\frac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x^2}} \overset{C}{=} \frac{1-0}{1+0} = 1$$

It follows that $y = 1$ is the horizontal asymptote of $f(x) = \dfrac{x^2-1}{x^2+1}$. The graph below confirms that:



EXAMPLES:

1. $$\lim_{x\to\pm\infty}\frac{2x^3-x+5}{x^3+x^2-1}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\dfrac{2x^3-x+5}{x^3}}{\dfrac{x^3+x^2-1}{x^3}}\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\dfrac{2x^3}{x^3}-\dfrac{x}{x^3}+\dfrac{5}{x^3}}{\dfrac{x^3}{x^3}+\dfrac{x^2}{x^3}-\dfrac{1}{x^3}}$$

$$\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{2-\dfrac{1}{x^2}+\dfrac{5}{x^3}}{1+\dfrac{1}{x}-\dfrac{1}{x^3}}\stackrel{C}{=}\frac{2-0+0}{1+0-0}=2$$

It follows that $y=2$ is the horizontal asymptote of the function $f(x)=\dfrac{2x^3-x+5}{x^3+x^2-1}$.

2. $$\lim_{x\to\pm\infty}\frac{3x+3x^2-7}{x+1-5x^2}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\dfrac{3x+3x^2-7}{x^2}}{\dfrac{x+1-5x^2}{x^2}}\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\dfrac{3x}{x^2}+\dfrac{3x^2}{x^2}-\dfrac{7}{x^2}}{\dfrac{x}{x^2}+\dfrac{1}{x^2}-\dfrac{5x^2}{x^2}}$$

$$\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\dfrac{3}{x}+3-\dfrac{7}{x^2}}{\dfrac{1}{x}+\dfrac{1}{x^2}-5}\stackrel{C}{=}\frac{0+3-0}{0+0-5}=-\frac{3}{5}$$

It follows that $y=-\dfrac{3}{5}$ is the horizontal asymptote of the function $f(x)=\dfrac{3x+3x^2-7}{x+1-5x^2}$.

3. $$\lim_{x\to\pm\infty}\frac{7x^2+10x+20}{x^3-10x^2-1}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\dfrac{7x^2+10x+20}{x^3}}{\dfrac{x^3-10x^2-1}{x^3}}\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\dfrac{7x^2}{x^3}+\dfrac{10x}{x^3}+\dfrac{20}{x^3}}{\dfrac{x^3}{x^3}-\dfrac{10x^2}{x^3}-\dfrac{1}{x^3}}$$

$$\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\dfrac{7}{x}+\dfrac{10}{x^2}+\dfrac{20}{x^3}}{1-\dfrac{10}{x}-\dfrac{1}{x^3}}\stackrel{C}{=}\frac{0+0+0}{1-0-0}=0$$

It follows that $y=0$ is the horizontal asymptote of the function $f(x)=\dfrac{7x^2+10x+20}{x^3-10x^2-1}$.

4. $$\lim_{x\to-\infty}\frac{11x^5+1}{4-x^4}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to-\infty}\frac{\dfrac{11x^5+1}{x^4}}{\dfrac{4-x^4}{x^4}}\stackrel{A}{=}\lim_{x\to-\infty}\frac{\dfrac{11x^5}{x^4}+\dfrac{1}{x^4}}{\dfrac{4}{x^4}-\dfrac{x^4}{x^4}}$$

$$\stackrel{A}{=}\lim_{x\to-\infty}\frac{11x+\dfrac{1}{x^4}}{\dfrac{4}{x^4}-1}\stackrel{C}{=}\left[\frac{-\infty}{-1}\right]=\infty \text{ (D.N.E)}$$

In the same way one can show that $\lim\limits_{x\to\infty}\dfrac{11x^5+1}{4-x^4}=-\infty$. It follows that the function $f(x)=\dfrac{11x^5+1}{4-x^4}$ does not have horizontal asymptotes.