Section 11.1 Limits

DEFINITION: We write

$$\boxed{\lim_{x\to a} f(x) = L}$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrary close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

EXAMPLE:

x approaches 2	x approaches 2	x approaches 2	x approaches 2	x does NOT approach 2	x does NOT approach 2	x approaches 2
1.9	2.1	2.4	2400000000⟵	2.1	2.4	2.4
1.99	2.01	1.98	198000⟵	-2.01⟵	1.98	1.98
1.999	2.001	2.004	200.04⟵	2.001	200.7⟵	200.7⟵
1.9999	2.0001	1.9995	19.995⟵	-2.0001⟵	1.9995	1.9995
1.99999	2.00001	2.00009	2.00009	2.00001	2.00009	2.00009
1.999999	2.000001	1.999999	1.999999	-2.000001⟵	9.87735⟵	9.87735⟵
1.9999999	2.0000001	1.9999997	1.9999997	2.0000001	1.9999997	1.9999997
1.99999999	2.00000001	2.00000004	2.00000004	-2.00000001⟵	20.5736732⟵	2.05736732
1.999999999	2.000000001	2.000000009	2.000000009	2.000000009	2.000000009	2.000000009
1.9999999999	2.0000000001	1.9999999991	1.9999999991	-1.9999999991⟵	1.9999999991	1.9999999991
1.99999999999	2.00000000001	2.00000000002	2.00000000002	2.00000000002	70.76456523⟵	2.00000000002
1.999999999999	2.000000000001	1.999999999994	1.999999999994	-1.999999999994⟵	1.999999999994	1.999999999994

EXAMPLES:

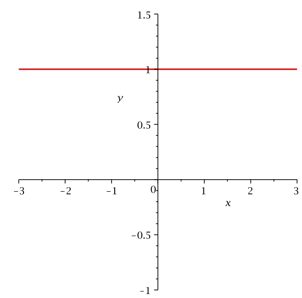
1. Let $f(x) = 1$ and $a = 2$. Estimate the value of $\lim_{x\to a} f(x)$.

Solution: We have

x	$f(x)$
1.9	1
1.99	1
1.999	1
1.9999	1
1.99999	1
1.999999	1
1.9999999	1
1.99999999	1

x	$f(x)$
2.1	1
2.01	1
2.001	1
2.0001	1
2.00001	1
2.000001	1
2.0000001	1
2.00000001	1

x	$f(x)$
2.4	1
1.98	1
2.004	1
1.9995	1
2.00009	1
1.999999	1
1.9999997	1
2.00000004	1



Looking at these tables and at the graph (or simply using common sense) one can conclude that $\lim_{x\to a} f(x) = \lim_{x\to 2} 1 = 1$.

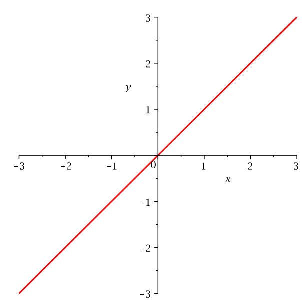
2. Let $f(x) = x$ and $a = 2$. Estimate the value of $\lim_{x\to a} f(x)$.

Solution: We have

x	$f(x)$
1.9	1.9
1.99	1.99
1.999	1.999
1.9999	1.9999
1.99999	1.99999
1.999999	1.999999
1.9999999	1.9999999
1.99999999	1.99999999

x	$f(x)$
2.1	2.1
2.01	2.01
2.001	2.001
2.0001	2.0001
2.00001	2.00001
2.000001	2.000001
2.0000001	2.0000001
2.00000001	2.00000001

x	$f(x)$
2.4	2.4
1.98	1.98
2.004	2.004
1.9995	1.9995
2.00009	2.00009
1.999999	1.999999
1.9999997	1.9999997
2.00000004	2.00000004



Looking at these tables and at the graph (or simply using common sense) one can conclude that $\lim_{x\to a} f(x) = \lim_{x\to 2} x = 2$.

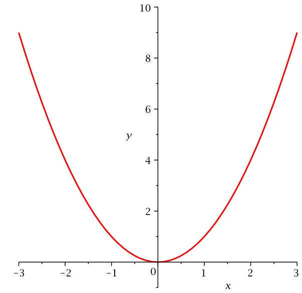
3. Estimate the value of $\lim_{x\to 2} x^2$.

Solution: We have

x	$f(x)$
1.9	3.610000000
1.99	3.960100000
1.999	3.996001000
1.9999	3.999600010
1.99999	3.999960000
1.999999	3.999996000
1.9999999	3.999999600
1.99999999	3.999999960

x	$f(x)$
2.1	4.410000000
2.01	4.040100000
2.001	4.004001000
2.0001	4.000400010
2.00001	4.000040000
2.000001	4.000004000
2.0000001	4.000000400
2.00000001	4.000000040

x	$f(x)$
2.4	4.840000000
1.98	4.161600000
2.004	3.980025000
1.9995	4.001200090
2.00009	3.999800002
1.999999	3.999968000
1.9999997	3.999998800
2.00000004	3.999999720



Looking at these tables and at the graph one can conclude that $\lim_{x\to 2} x^2 = 4$. Note that the value of $f(x) = x^2$ at 2 is also 4.

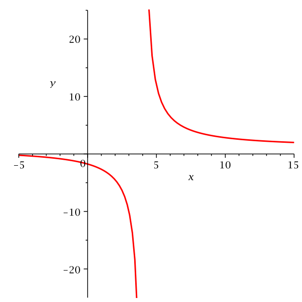
4. Estimate the value of $\lim_{x\to 3} \frac{x+7}{x-4}$.

Solution: We have

x	$f(x)$
2.9	-9.000000000
2.99	-9.891089109
2.999	-9.989010989
2.9999	-9.998900110
2.99999	-9.999890001
2.999999	-9.999989000
2.9999999	-9.999998900
2.99999999	-9.999999890

x	$f(x)$
3.1	-11.22222222
3.01	-10.11111111
3.001	-10.01101101
3.0001	-10.00110011
3.00001	-10.00011000
3.000001	-10.00001100
3.0000001	-10.00000110
3.00000001	-10.00000011

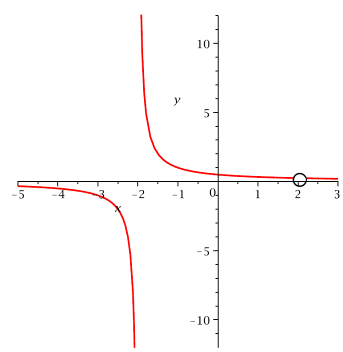
x	$f(x)$
2.9	-5.111111111
3.02	-10.34020619
3.006	-9.956175299
3.0002	-9.996700990
3.00004	-10.00077005
2.999998	-9.999978000
2.9999994	-9.999990100
2.99999996	-9.999999560



Looking at these tables and at the graph one can conclude that $\lim_{x\to 3} \frac{x+7}{x-4} = -10$. Note that the value of $f(x) = \frac{x+7}{x-4}$ at 3 is also -10.

5. Estimate the value of $\lim_{x\to 2} \frac{x-2}{x^2-4}$.

Solution: Note that $f(x) = \frac{x-2}{x^2-4}$ does not exist at 2. We have



x	$f(x)$
1.9	0.2564102564
1.99	0.2506265664
1.999	0.2500625156
1.9999	0.2500062502
1.99999	0.2500000000
1.999999	0.2500000000
1.9999999	0.2500000000
1.99999999	0.2500000000

x	$f(x)$
2.1	0.2439024390
2.01	0.2493765586
2.001	0.2499375156
2.0001	0.2499937502
2.00001	0.2500000000
2.000001	0.2500000000
2.0000001	0.2500000000
2.00000001	0.2500000000

x	$f(x)$
1.2	0.2564102564
1.93	0.2469135802
2.008	0.2503128911
2.0007	0.2499687539
2.00009	0.2500006250
2.000009	0.2499995000
2.0000006	0.2499999500
1.999999970	0.2499999994

Looking at these tables and at the graph one can conclude that $\lim_{x\to 2} \frac{x-2}{x^2-4} = 0.25$.

REMARK: Note that if $x \neq \pm 2$, then

$$\frac{x-2}{x^2-4} = \frac{x-2}{x^2-2^2} = \frac{(x-2)\cdot 1}{(x-2)(x+2)} = \frac{1}{x+2}$$

therefore $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{1}{x+2}$, which explains the above answer.

Calculating Limits

LIMIT LAWS: Let a be a real number and let f and g be functions such that

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0$
5. For any real number r for which $\left[\lim_{x\to a} f(x)\right]^r$ exists, $\lim_{x\to a}[f(x)]^r = \left[\lim_{x\to a} f(x)\right]^r$

EXAMPLES:

1. $\lim_{x\to 3}(4x^2-3x+5) = \lim_{x\to 3}(4x^2) - \lim_{x\to 3}(3x) + \lim_{x\to 3} 5 = 4\lim_{x\to 3} x^2 - 3\lim_{x\to 3} x + \lim_{x\to 3} 5$

$$= 4\cdot 3^2 - 3\cdot 3 + 5 = 32$$

2. $\lim_{x\to -1} \frac{x^2-x+1}{2-x} = \frac{\lim_{x\to -1}(x^2-x+1)}{\lim_{x\to -1}(2-x)} = \frac{\lim_{x\to -1} x^2 - \lim_{x\to -1} x + \lim_{x\to -1} 1}{\lim_{x\to -1} 2 - \lim_{x\to -1} x}$

$$= \frac{(-1)^2-(-1)+1}{2-(-1)} = 1$$

DIRECT SUBSTITUTION PROPERTY: If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

EXAMPLES:

1′. $\lim_{x\to 3}(4x^2-3x+5) \overset{DSP}{=} 4\cdot 3^2 - 3\cdot 3 + 5 = 32$

2′. $\lim_{x\to -1} \frac{x^2-x+1}{2-x} \overset{DSP}{=} \frac{(-1)^2-(-1)+1}{2-(-1)} = 1$

3. $\lim_{x\to 2} \frac{x-2}{x^2-4} = \left[\frac{0}{0}\right] \overset{A}{=} \lim_{x\to 2} \frac{x-2}{x^2-2^2} \overset{A}{=} \lim_{x\to 2} \frac{(x-2)\cdot 1}{(x-2)(x+2)} \overset{A}{=} \lim_{x\to 2} \frac{1}{x+2} \overset{DSP}{=} \frac{1}{2+2} = \frac{1}{4}$

In short,

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$$

4. $\lim_{x\to -1} \frac{x^2+3x+2}{x^2+9x+8} = \left[\frac{0}{0}\right] \overset{A}{=} \lim_{x\to -1} \frac{(x+1)(x+2)}{(x+1)(x+8)} \overset{A}{=} \lim_{x\to -1} \frac{x+2}{x+8} \overset{DSP}{=} \frac{-1+2}{-1+8} = \frac{1}{7}$

5. $\lim_{x\to -2} \frac{x^2+3x+2}{x^2-1} \overset{DSP}{=} \frac{(-2)^2+3\cdot(-2)+2}{(-2)^2-1} = \frac{0}{3} = 0$

6. $\lim_{x\to 3} \frac{x^2-7x+12}{x^2-6x+9} = \left[\frac{0}{0}\right] \overset{A}{=} \lim_{x\to 3} \frac{(x-3)(x-4)}{(x-3)^2} \overset{A}{=} \lim_{x\to 3} \frac{x-4}{x-3}$ DOES NOT EXIST

7. $$\lim_{h\to 0} \frac{(5+h)^2-25}{h} = \left[\frac{0}{0}\right] \overset{A}{=} \lim_{h\to 0} \frac{25+10h+h^2-25}{h} \overset{A}{=} \lim_{h\to 0} \frac{10h+h^2}{h}$$
$$\overset{A}{=} \lim_{h\to 0} \frac{h(10+h)}{h} \overset{A}{=} \lim_{h\to 0}(10+h) \overset{DSP}{=} 10$$

8. $$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \left[\frac{0}{0}\right] \overset{A}{=} \lim_{x\to 4} \frac{\sqrt{x}-2}{(\sqrt{x})^2-2^2} \overset{A}{=} \lim_{x\to 4} \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} \overset{A}{=} \lim_{x\to 4} \frac{1}{\sqrt{x}+2} \overset{LL}{=} \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

or

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \left[\frac{0}{0}\right] \overset{A}{=} \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} \overset{A}{=} \lim_{x\to 4} \frac{(\sqrt{x})^2-2^2}{(x-4)(\sqrt{x}+2)}$$
$$\overset{A}{=} \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)}$$
$$\overset{A}{=} \lim_{x\to 4} \frac{1}{\sqrt{x}+2}$$
$$\overset{LL}{=} \frac{1}{\sqrt{4}+2}$$
$$= \frac{1}{4}$$

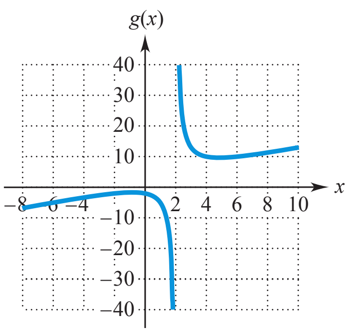
Existence of Limits

It is possible that $\lim_{x \to a} f(x)$ may not exist; that is, there may be no number L satisfying the definition of $\lim_{x \to a} f(x) = L$. This can happen in many ways, two of which are illustrated next.

1. Let

$$g(x) = \frac{x^2 + 4}{x - 2}$$

We show that $\lim_{x \to 2} g(x)$ does not exist. Indeed, we first note that the quotient property cannot be used, since $\lim_{x \to 2}(x - 2) = 0$, and the limit theorem does not apply because $x^2 + 4$ does not factor. So we use the following table and the graph of g:



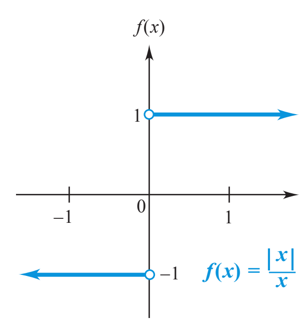
	x approaches 2 from the left →				2	← *x* approaches 2 from the right		
x	1.8	1.9	1.99	1.999	2	2.001	2.01	2.05
***g*(*x*)**	−36.2	−76.1	−796	−7996		8004	804	164
		g(*x*) gets smaller and smaller →				← *g*(*x*) gets larger and larger		

The table and the graph both show that as x approaches 2 from the left, $g(x)$ gets smaller and smaller, but as x approaches 2 from the right, $g(x)$ gets larger and larger. Since $g(x)$ does not get closer and closer to a single real number as x approaches 2 from either side,

$$\lim_{x \to 2} \frac{x^2 + 4}{x - 2} \quad \text{does not exist}$$

2. We show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. In fact, the function $f(x) = \frac{|x|}{x}$ is not defined when $x = 0$. Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Consequently, when $x > 0$,

$$f(x) = \frac{|x|}{x} = \frac{x}{x} = 1$$

and when $x < 0$,

$$f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1$$

The graph of f is shown in the Figure above. As x approaches 0 from the right, x is always positive, and the corresponding value of $f(x)$ is 1. But as x approaches 0 from the left, x is always negative, and the corresponding value of $f(x)$ is -1. Thus, as x approaches 0 from both sides, the corresponding values of $f(x)$ do not get closer and closer to a *single* real number. Therefore, the limit does not exist.

REMARK: The behavior of this function near $x = 0$ can be described by *one-sided limits*, which are discussed in the next section.

The function f whose graph is shown in the Figure below illustrates various facts about limits that were discussed earlier in this section.