

Section 6.7 Joint Distribution of Functions of Random Variables

Let X_1 and X_2 be jointly continuous random variables with joint probability density function f_{X_1, X_2} . It is sometimes necessary to obtain the joint distribution of the random variables Y_1 and Y_2 , which arise as functions of X_1 and X_2 . Specifically, suppose that $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ for some functions g_1 and g_2 .

Assume that the functions g_1 and g_2 satisfy the following conditions:

1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2 with solutions given by, say, $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.
2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the following 2×2 determinant

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \equiv \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \neq 0$$

at all points (x_1, x_2) .

Under these two conditions it can be shown that the random variables Y_1 and Y_2 are jointly continuous with joint density function given by

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) |J(x_1, x_2)|^{-1} \quad (7.1)$$

where $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.

A proof of Equation (7.1) would proceed along the following lines:

$$P\{Y_1 \leq y_1, Y_2 \leq y_2\} = \iint_{\substack{(x_1, x_2): \\ g_1(x_1, x_2) \leq y_1 \\ g_2(x_1, x_2) \leq y_2}} f_{X_1, X_2}(x_1, x_2) dx_1 dx_2 \quad (7.2)$$

The joint density function can now be obtained by differentiating Equation (7.2) with respect to y_1 and y_2 . That the result of this differentiation will be equal to the right-hand side of Equation (7.1) is an exercise in advanced calculus whose proof will not be presented in this book.

Example 7a. Let X_1 and X_2 be jointly continuous random variables with probability density function f_{X_1, X_2} . Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1, X_2} .

Solution Let $g_1(x_1, x_2) = x_1 + x_2$ and $g_2(x_1, x_2) = x_1 - x_2$. Then

$$J(x_1, x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

Also, as the equations $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ have as their solution $x_1 = (y_1 + y_2)/2$, $x_2 = (y_1 - y_2)/2$, it follows from Equation (7.1) that the desired density is

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1, X_2} \left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2} \right)$$

For instance, if X_1 and X_2 are independent, uniform $(0, 1)$ random variables, then

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2} & 0 \leq y_1 + y_2 \leq 2, 0 \leq y_1 - y_2 \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

or if X_1 and X_2 were independent, exponential random variables with respective parameters λ_1 and λ_2 , then

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \frac{\lambda_1 \lambda_2}{2} \exp \left\{ -\lambda_1 \left(\frac{y_1 + y_2}{2} \right) - \lambda_2 \left(\frac{y_1 - y_2}{2} \right) \right\} & y_1 + y_2 \geq 0, y_1 - y_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally, if X_1 and X_2 are independent unit normal random variables,

$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= \frac{1}{4\pi} e^{-[(y_1 + y_2)^2/8 + (y_1 - y_2)^2/8]} \\ &= \frac{1}{4\pi} e^{-(y_1^2 + y_2^2)/4} \\ &= \frac{1}{\sqrt{4\pi}} e^{-y_1^2/4} \frac{1}{\sqrt{4\pi}} e^{-y_2^2/4} \end{aligned}$$

Thus, not only do we obtain (in agreement with Proposition 3.2) that both $X_1 + X_2$ and $X_1 - X_2$ are normal with mean 0 and variance 2, we also obtain the interesting result that these two random variables are independent. (In fact, it can be shown that if X_1 and X_2 are independent random variables having a common distribution function F , then $X_1 + X_2$ will be independent of $X_1 - X_2$ if and only if F is a normal distribution function.) ■