Section 6.5 Conditional Distributions: Continuous Case

Let X and Y be continuous random variables with joint density $f(x, y)$. We want to define the density of X conditioned on $Y = y$. We motivate it in the following way:

$$\lim_{\triangle y \to 0} P\{x \leq X \leq x + \triangle x \mid y \leq Y \leq y + \triangle y\}$$

$$= \lim_{\triangle y \to 0} \frac{P\{x \leq X \leq x + \triangle x, y \leq Y \leq y + \triangle y\}}{P\{y \leq Y \leq y + \triangle y\}} \approx \lim_{\triangle y \to 0} \frac{f(x,y)\triangle x \triangle y}{f_Y(y)\triangle y} = \frac{f(x,y)}{f_Y(y)}\triangle x$$

Therefore, we take

$$\boxed{f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}}$$

when $f_Y(y) > 0$. Now we can compute conditional probabilities. For $A \subset \mathbb{R}$ we have

$$\boxed{P\{X \in A \mid Y = y\} = \int\limits_A f_{X|Y}(x|y)dx}$$

It is obvious how to define the conditional distribution function

$$\boxed{F_{X|Y}(t|y) = P\{X \leq t \mid Y = y\} = \int\limits_{-\infty}^{t} f_{X|Y}(x|y)dx}$$

REMARK: We have given workable expressions for conditional probabilities even though the event we are conditioning on, $\{Y = y\}$, has a probability of zero!

EXAMPLE: Suppose that the joint density of X and Y is

$$f(x,y) = \frac{12}{5}x(2 - x - y) \quad \text{for } 0 < x < 1 \text{ and } 0 < y < 1$$

and zero otherwise. Find $f_{X|Y}(x|y)$ for $y \in (0, 1)$.

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Solution: we have

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{f(x,y)}{\int\limits_{-\infty}^{\infty}f(x,y)dx}=\frac{x(2-x-y)}{\int\limits_0^1 x(2-x-y)dx}=\frac{x(2-x-y)}{2/3-y/2}=\frac{6x(2-x-y)}{4-3y}$$

Thus, for example,

$$f_{X|Y}(x|1/3)=2x(5/3-x)=\frac{10}{3}x-2x^2 \quad \text{and} \quad \int\limits_0^1 f_{X|Y}(x|(1/3))dx=1$$

EXAMPLE: Suppose

$$f(x,y)=(x+y)/4, \quad 0<x<y<2$$

Find the conditional density of X given that $Y=y$ and the conditional density of Y given that $X=x$. Specify the range of values of the variables.

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Find the conditional density of X given that $Y=y$ and the conditional density of Y given that $X=x$. Specify the range of values of the variables.

Solution: First, for $0<x<y<2$ we have

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{f(x,y)}{\int\limits_{-\infty}^{\infty}f(x,y)dx}=\frac{\frac{1}{4}(x+y)}{\int\limits_{0}^{y}\frac{1}{4}(x+y)dx}=\frac{\frac{1}{4}(x+y)}{\frac{3}{8}y^2}=\frac{2}{3}\cdot\frac{x+y}{y^2}$$

For example,

$$f_{X|Y}(x|1)=\frac{2}{3}(x+1), \quad x\in(0,1)$$

$$f_{X|Y}(x|3/2)=\frac{2}{3}\cdot\frac{x+3/2}{9/4}=\frac{8}{27}\left(x+3/2\right), \quad x\in(0,3/2)$$

Note that for $x<y<2$ we have

$$\int\limits_{0}^{y}f_{X|Y}(x|y)dx=\frac{2}{3}\int\limits_{0}^{y}\left(\frac{1}{y^2}x+\frac{1}{y}\right)dx=\frac{2}{3}\left(\frac{1}{2y^2}x^2+\frac{1}{y}x\right)\Bigg|_{x=0}^{y}=1$$

Also,

$$P\{X<1 \mid Y=3/2\}=\int\limits_{0}^{1}f_{X|Y}(x|3/2)dx=\int\limits_{0}^{1}\frac{8}{27}(x+3/2)dx=\frac{4}{27}x^2+\frac{4}{9}x\Bigg|_{0}^{1}=\frac{16}{27}$$

Next, for $0<x<y<2$ we have

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{f(x,y)}{\int\limits_{-\infty}^{\infty}f(x,y)dy}=\frac{\frac{1}{4}(x+y)}{\int\limits_{x}^{2}\frac{1}{4}(x+y)dy}=\frac{\frac{1}{4}(x+y)}{\frac{1}{8}(4x+4-3x^2)}=2\cdot\frac{x+y}{4x+4-3x^2}$$

For example,

$$f_{Y|X}(y|1/2)=\frac{8}{3}\left(y+\frac{1}{2}\right)$$

REMARK: Obviously, If X and Y are independent, then

$$\boxed{f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{f_X(x)f_Y(y)}{f_Y(y)}=f_X(x)}$$