Section 6.5 Conditional Distributions: Continuous Case

Let $X$ and $Y$ be continuous random variables with joint density $f(x, y)$. We want to define the density of $X$ conditioned on $Y = y$. We motivate it in the following way:

$$\lim_{\Delta y \to 0} P\{x \leq X \leq x + \Delta x \mid y \leq Y \leq y + \Delta y\} = \lim_{\Delta y \to 0} \frac{P\{x \leq X \leq x + \Delta x, y \leq Y \leq y + \Delta y\}}{P\{y \leq Y \leq y + \Delta y\}} \approx \lim_{\Delta y \to 0} \frac{f(x, y) \Delta x \Delta y}{f_Y(y) \Delta y} = \frac{f(x, y)}{f_Y(y)} \Delta x$$

Therefore, we take

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

when $f_Y(y) > 0$. Now we can compute conditional probabilities. For $A \subset \mathbb{R}$ we have

$$P\{X \in A \mid Y = y\} = \int_A f_{X|Y}(x|y)dx$$

It is obvious how to define the conditional distribution function

$$F_{X|Y}(t|y) = P\{X \leq t \mid Y = y\} = \int_{-\infty}^{t} f_{X|Y}(x|y)dx$$

REMARK: We have given workable expressions for conditional probabilities even though the event we are conditioning on, \{Y = y\}, has a probability of zero!

EXAMPLE: Suppose that the joint density of $X$ and $Y$ is

$$f(x, y) = \frac{12}{5} x(2 - x - y) \quad \text{for } 0 < x < 1 \text{ and } 0 < y < 1$$

and zero otherwise. Find $f_{X|Y}(x|y)$ for $y \in (0, 1)$. 
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Solution: we have

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{x(2 - x - y)}{1} = \frac{x(2 - x - y)}{2/3 - y/2} = \frac{6x(2 - x - y)}{4 - 3y}$$

Thus, for example,

$$f_{X|Y}(x|1/3) = 2x(5/3 - x) = \frac{10}{3}x - 2x^2 \quad \text{and} \quad \int_0^1 f_{X|Y}(x|1/3)dx = 1$$

EXAMPLE: Suppose

$$f(x, y) = (x + y)/4, \quad 0 < x < y < 2$$

Find the conditional density of $X$ given that $Y = y$ and the conditional density of $Y$ given that $X = x$. Specify the range of values of the variables.
EXAMPLE: Suppose
\[ f(x, y) = (x + y)/4, \quad 0 < x < y < 2 \]
Find the conditional density of \( X \) given that \( Y = y \) and the conditional density of \( Y \) given that \( X = x \). Specify the range of values of the variables.

Solution: First, for \( 0 < x < y < 2 \) we have
\[
 f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{1}{4}(x + y)}{\int_{-\infty}^{y} \frac{1}{4}(x + y) dx} = \frac{\frac{1}{4}(x + y)}{\frac{3}{8}y^2} = \frac{2}{3} \cdot \frac{x + y}{y^2}
\]
For example,
\[
 f_{X|Y}(x|1) = \frac{2}{3}(x + 1), \quad x \in (0, 1)
\]
\[
 f_{X|Y}(x|3/2) = \frac{2}{3} \cdot \frac{x + 3/2}{9/4} = \frac{8}{27}(x + 3/2), \quad x \in (0, 3/2)
\]
Note that for \( x < y < 2 \) we have
\[
 \int_{0}^{y} f_{X|Y}(x|y) dx = \frac{2}{3} \int_{0}^{y} \left( \frac{1}{y^2} x + \frac{1}{y} \right) dx = \frac{2}{3} \left( \frac{1}{2y^2}x^2 + \frac{1}{y} x \right) \bigg|_{x=0}^{y} = 1
\]
Also,
\[
 P\{X < 1 \mid Y = 3/2\} = \int_{0}^{1} f_{X|Y}(x|3/2) dx = \int_{0}^{1} \frac{8}{27}(x + 3/2) dx = \left. \frac{4}{27}x^2 + \frac{4}{9}x \right|_{0}^{1} = \frac{16}{27}
\]
Next, for \( 0 < x < y < 2 \) we have
\[
 f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{\frac{1}{4}(x + y)}{\int_{-\infty}^{\infty} \frac{1}{4}(x + y) dy} = \frac{\frac{1}{4}(x + y)}{\frac{1}{8}(4x + 4 - 3x^2)} = 2 \cdot \frac{x + y}{4x + 4 - 3x^2}
\]
For example,
\[
 f_{Y|X}(y|1/2) = \frac{8}{3} \left( y + \frac{1}{2} \right)
\]
REMARK: Obviously, If \( X \) and \( Y \) are independent, then
\[
 f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)
\]