Section 2.2 Sample Space and Events

We consider an experiment whose outcome is not predictable with certainty. However, we suppose that the set of all possible outcomes is known.

DEFINITION: The set of all possible outcomes of an experiment is the *sample space* of an experiment, and is denoted by S. Subsets of S are called *events*. Elements of S, $x \in S$, are called *outcomes*.

EXAMPLES:

1. A coin is tossed twice and the outcome of each is recorded. Then,

$$S = \{(H,\ H),\ (H,\ T),\ (T,\ H),\ (T,\ T)\}$$

The event that the second toss was a Head is the subset

$$E = \{(H,\ H),\ (T,\ H)\}$$

2. Consider 3 light-bulbs. Our experiment consists of finding out which light-bulb burns out first, and how long (in hours) it takes for this to happen.

$$S = \{(i,\ t) :\ i \in \{1, 2, 3\},\ t\ \geq 0\}$$

i tells you which one burns out, and t gives how long it lasted, in hours. The event that the 2nd bulb burns out first, and it lasts less than 3 hours is the set

$$E = \{(2,\ t) :\ t\ < 3\}$$

3. You roll a four sided die until a 4 comes up. The event you are interested in is getting a three with the first two rolls.

$$S = \{\ a_1 a_2 \ldots a_n \mid n\ \geq 1,\ a_n = 4,\ a_i \in \{1, 2, 3\} \text{ for } i \neq n\}$$

$$E = \{\ 33 a_3 a_4 \ldots a_n \mid n\ \geq 3,\ a_n = 4,\ a_i \in \{1, 2, 3\} \text{ for } i \neq n\}$$

DEFINITION: The *union* of E and F, denoted $E \cup F$, consists of those outcomes that are in either E or F (or both). (we write $x \in E \cup F$ if $x \in E$ *or* $x \in F$.)

EXAMPLE: If $E = \{(H,\ H),\ (H,\ T)\}$ (Head on first toss) and $F = \{(T,\ T),\ (H,\ T)\}$ (tail on the second toss). Then,

$$E \cup F = \{(H,\ H),\ (H,\ T),\ (T,\ T)\}$$

only missing $(T,\ H)$.

DEFINITION: The *intersection* of E and F, denoted EF or $E \cap F$, consists of those events that are in both E and F. (we write $x \in EF$ if $x \in E$ *and* $x \in F$.)

EXAMPLE: If $E = \{(H,\ H),\ (H,\ T)\}$ (Head on first toss) and $F = \{(T,\ T),\ (H,\ T)\}$ (tail on the second toss). Then

$$EF = \{(H,\ T)\}$$

DEFINITION: The *empty set* $\{\} = \emptyset$ is the set consisting of nothing.

DEFINITION: Two sets are *mutually exclusive* if $EF = \emptyset$. A set of sets, $\{E_1,\ E_2,\ \ldots,\ \}$ are mutually exclusive if $E_iE_j = \emptyset$ for all $i \neq j$.

EXAMPLE: Let $S = \{(i,\ j) \mid i,\ j \in \{1,\ \ldots,\ 6\}\}$ be the outcome from two rolls of a die. Let E be those such that $i+j = 6$ (i.e. $E = \{(1,5),\ (2,\ 4),\ \ldots\}$) and F be those such that $i+j = 7$ (i.e. $F = \{(1,6),\ (2,\ 5),\ \ldots\}$). Then $EF = \emptyset$ and they are mutually exclusive.

The definition for the union and intersection of a sequence of events $E_1,\ E_2,\ \ldots$ are similar:

1. $\bigcup\limits_{n=1}^{\infty} E_n$ is the event consisting of those outcomes that are in at least one E_n, for $n = 1, 2,\ \ldots$.
2. $\bigcap\limits_{n=1}^{\infty} E_n$ is the event consisting of those outcomes that are in each E_n for $n = 1, 2,\ \ldots$.

DEFINITION: The *complement* of E, denoted E^c, consists of those outcomes that are not in E. Note that $S^c = \emptyset$ and that $(E^c)^c = E$.

DEFINITION: E is a *subset* of F if $x \in E$ implies $x \in F$. Notation: $E \subset F$.

DEFINITION: E and F are *equal*, denoted $E = F$, if $E \subset F$ and $F \subset E$.

Important Set Relations

Commutative laws:

$$E \cup F = F \cup E, \quad EF = FE$$

Associative laws:

$$E \cup (F \cup G) = (E \cup F) \cup G, \quad E(FG) = (EF)G$$

Distributive laws:

$$(EF) \cup H = (E \cup H)(F \cup H), \quad (E \cup F)H = (EH) \cup (FH)$$

De Morgan's laws:

$$\left(\bigcup_{i=1}^{\infty} E_i\right)^c = \bigcap_{i=1}^{\infty} E_i^c$$

Note that if $E_i = \emptyset$ for $i \geq 3$, we have $(E \cup F)^c = E^c \cap F^c$.

$$\left(\bigcap_{i=1}^{\infty} E_i\right)^c = \bigcup_{i=1}^{\infty} E_i^c$$

Note that if $E_i = \emptyset$ for $i \geq 3$, we have $(E \cap F)^c = E^c \cup F^c$.

Proof: Let $x \in \left(\bigcap_{i=1}^{\infty} E_i\right)^c$. Then, $x \notin \bigcap_{i=1}^{\infty} E_i$. Thus, $x \notin E_i$ for at least one E_i. So $x \in E_i^c$ for that E_i and so

$$x \in \bigcup_{i=1}^{\infty} E_i^c \quad \Longrightarrow \quad \left(\bigcap_{i=1}^{\infty} E_i\right)^c \subset \bigcup_{i=1}^{\infty} E_i^c$$

Let $x \in \bigcup_{i=1}^{\infty} E_i^c$. Then $x \in E_i^c$ for at least one E_i and so $x \notin E_i$ and, hence, $x \notin \bigcap_{i=1}^{\infty} E_i$. Therefore

$$x \in \left(\bigcap_{i=1}^{\infty} E_i\right)^c \quad \text{and} \quad \bigcup_{i=1}^{\infty} E_i^c \subset \left(\bigcap_{i=1}^{\infty} E_i\right)^c \ \blacksquare$$

Section 2.3 Axioms of Probability

Basic Question: What do we mean by the probability of an event? Good idea: Relative frequency interpretation. Let $n(E)$ be the number of times that E occurs during n performances of the experiment. Finally, define

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

EXAMPLES:

1. Rolling a fair die. Will find P (6 is rolled) $\approx 1/6$.

2. Flipping a fair coin twice. Will find P (first heads, then tails) $\approx 1/4$.

REMARKS:

1. No reason to believe that $\lim_{n\to\infty} \frac{n(E)}{n}$ actually converges, this has to be an assumption.

2. Notions that do not have repeatability do not have meaning: for example, probability of rain tomorrow or guilt/innocence in criminal cases.

We will take a different route. An axiomatic approach. We will eventually show that after assuming three basic axioms, the limit above does exist (law of large numbers, chapter 8).

Axioms of Probability

Let S be a sample space for some experiment. For each event $E \subset S$, we assume that a number $P(E)$ is defined and satisfies the following three axioms:

Axiom 1: $0 \le P(E) \le 1$.

Axiom 2: $P(S) = 1$.

Axiom 3: If $\{E_i\}$ are mutually exclusive ($E_i \cap E_j = \emptyset$ if $i \neq j$), then $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$.

We call $P(A)$ the probability of E.

THEOREM: $P(\emptyset) = 0$.

Proof: We have $S = S \cup \bigcup_{i=2}^{\infty} \emptyset$. Note that $\{S, \emptyset, \emptyset, \ldots\}$ are mutually exclusive. Thus,

$$P(S) = P\left(S \cup \bigcup_{i=2}^{\infty} \emptyset\right) = P(S) + \sum_{i=2}^{\infty} P(\emptyset)$$

Thus, $P(\emptyset) = 0$. ■

THEOREM: *Let $A_1, \ldots, A_n$ be mutually exclusive (but finite). Then,*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Proof: Use previous result and take sum to infinity with the rest being the empty set. ■

EXAMPLES:

1. Consider flipping a coin and recording the outcome. Then $S = \{H, T\}$. Need a probability. Assume fair coin. Then

$$P(\{H\}) = 1/2$$
$$P(\{T\}) = 1/2$$
$$P(\{H, T\}) = 1$$

Note that

$$P(\{H\} \cup \{T\}) = 1 = 1/2 + 1/2 = P(\{H\}) + P(\{T\})$$

This is not the only probability possible for this experiment/sample space. Could have an unfair coin.

$$P(\{H\}) = p, \quad P(\{T\}) = 1 - p, \quad P(\{H, T\}) = 1$$

2. Rolling a die. Assuming that the die is fair, $P(\{1\}) = P(\{2\}) = \ldots = P(\{6\}) = 1/6$. Probability of rolling an even number is

$$P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/6 + 1/6 + 1/6 = 1/2$$

Section 2.4 Some Simple Propositions

THEOREM: For any event E we have

$$P(E^c) = 1 - P(E)$$

Proof: For any set, E and E^c are mutually exclusive. Therefore,

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c) \ ■$$

EXAMPLE: Roll two fair dice. Sample space

$$S = \{(i,\ j) :\ 1\ \leq i,\ j \leq 6\}$$

Assume fair dice, so probability of each event is 1/36. Let A be the event that the sum of the rolls equals $1, 2, 3, 5, 6, \ldots,$ or 12. Hard to calculate. But

$$P(A^c) = P(sum\ is\ 4)\ = P(\{(1,3),\ (2,2),\ (3,1)\}) = 3/36$$

So, $P(A) = 1 - 3/36 = 33/36 = 11/12$.

THEOREM: If $A \subset B$, then $P(A) \leq P(B)$.

Proof: Because $A \subset B$, we can write B as

$$B\ = A \cup (A^c B)$$

These are mutually exclusive and so

$$P(B)\ = P(A) + P(A^c B)$$

$P(A^c B) \geq 0$ and so the result is shown. ■

THEOREM: $P(E \cup F) = P(E) + P(F) - P(EF)$.

Proof: Note that $E \cup F = E \cup E^c F$, and these are mutually exclusive. Thus,

$$P(E \cup F) = P(E) + P(E^c F)$$

Also, $F = EF \cup E^c F$ (which are mutually exclusive) and so

$$P(F) = P(EF) + P(E^c F)$$

Combining the two equations shows

$$P(E \cup F) = P(E) + P(E^c F) = P(E) + P(F) - P(EF)\ ■$$

THEOREM: For any two sets E and F we have

$$P(F) = P(EF) + P(E^c F)$$

EXAMPLE: You believe the following: with probability 0.5 it will snow today. With probability 0.3 it will snow tomorrow. With probability 0.2 it will snow both days. What is the probability that it will not snow either day?

EXAMPLE: You believe the following: with probability 0.5 it will snow today. With probability 0.3 it will snow tomorrow. With probability 0.2 it will snow both days. What is the probability that it will not snow either day?

Solution: Let $E,\ F$ be the events that it snows today/tomorrow, resp. Then,

$$P(E) = 0.5,\ P(F) = 0.3, \quad \text{and} \quad P(EF) = 0.2$$

We need $P(E^cF^c)$. Not snowing, is the complement of snowing at least once, $E \cup F$, and so,

$$P(E^cF^c) = 1 - P(E \cup F) = 1 - (P(E) + P(F) - P(EF)) = 1 - (0.5 + 0.3 - 0.2) = 1 - 0.6 = 0.4$$

We can generalize the previous theorem:

THEOREM: For any three sets $A,\ B,\ C$, we have

$$P(A \cup B \ \cup C) = P(A) + P(B) \ + P(C) - P(AB) \ - P(BC) - P(AC) + P(ABC)$$

Proof: Proof by Venn diagram. ■

THEOREM (inclusion-exclusion identity): Let $E_1,\ E_2,\ \ldots,\ E_n$ be a sequence of sets. Then

$$\begin{aligned} P(E_1 \cup E_2 \cup \ldots \cup E_n) = & \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1}E_{i_2}) + \ldots \\ & + (-1)^{r+1} \sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1}E_{i_2}\ldots E_{i_r}) \\ & + \ldots + (-1)^{n+1} P(E_1E_2\ldots E_n) \end{aligned}$$

the summation $\displaystyle\sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1}E_{i_2}\ldots E_{i_r})$ is taken over the $\binom{n}{r}$ possible sets of size r of the set $\{1,\ 2,\ \ldots,\ n\}$.

Proof: Can be done by induction. ■

Section 2.5 Sample Spaces Having Equally Likely Outcomes

We have already seen that in many experiments it is natural to assume all outcomes are equally likely. That is, if $S = \{1, 2,\ \ldots,\ N\}$ then,

$$P(\{1\}) = P(\{2\}) = \ldots = P(\{N\}) = \frac{1}{N}$$

It then follows by Axiom 3 that for any event E

$$P(E) = \frac{\#\text{ elements of E}}{\#\text{ elements in S}}$$

EXAMPLES:

1. A number is chosen at random from $S = \{1, \ldots, 520\}$. What is the probability that it is divisible by 5 or 7?

2. If 3 balls are randomly drawn from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two are black?

3. A committee of 6 is to be chosen from 15 people. There are 5 Americans, 3 Canadians, and 7 Europeans in the group. Assuming the committee is randomly chosen, what is the probability that the committee will consist of 2 Americans, 1 Canadian, and 3 Europeans?

EXAMPLES:

1. A number is chosen at random from $S = \{1, \ldots, 520\}$. What is the probability that it is divisible by 5 or 7?

Solution: Assume each number has a probability of $1/520$ of being chosen. Let A be the event that it is divisible by 5. Let B be the event that it is divisible by 7. We have $520/5 = 104$, so $P(A) = 104/520$. Similarly, $520/7 = 74 + 2/7$, so $P(B) = 74/520$. Want $P(A \cup B)$. So, by the theorem, we need $P(A \cap B)$. $x \in A \cap B$ if and only if divisible by 35. $520/35 = 14 + 30/35$. So $P(A \cap B) = 14/520$. Thus,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 104/520 + 74/520 - 14/520 = 164/520$$

2. If 3 balls are randomly drawn from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two are black?

Solution 1: Think of order being relevant. There are $11 \cdot 10 \cdot 9 = 990$ total possible choices of a first, second, third ball. There are $6 \cdot (5 \cdot 4)$ ways to first choose a white, then two black, there are $5 \cdot 6 \cdot 4$ ways to choose black, white, black, and $5 \cdot 4 \cdot 6$ ways to choose two black, then white. So there are a total of $3 \cdot 6 \cdot 5 \cdot 4 = 360$ ways to choose one white and two black. Thus, the answer is

$$\frac{360}{990} = \frac{4}{11}$$

Solution 2: Now suppose order is not important. There are now $\binom{11}{3}$ ways to choose three balls. There are $\binom{6}{1}$ and $\binom{5}{2}$ ways to choose one white and two black balls, respectively. Therefore, the probability is

$$\frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{6 \cdot \dfrac{5!}{3!2!}}{\dfrac{11!}{8!3!}} = \frac{\dfrac{5!}{2!}}{\dfrac{11 \cdot 10 \cdot 9}{3!}} = \frac{5 \cdot 4 \cdot 3 \cdot 3 \cdot 2}{11 \cdot 10 \cdot 9} = \frac{4}{11}$$

3. A committee of 6 is to be chosen from 15 people. There are 5 Americans, 3 Canadians, and 7 Europeans in the group. Assuming the committee is randomly chosen, what is the probability that the committee will consist of 2 Americans, 1 Canadian, and 3 Europeans?

Solution: There are a total of $\binom{15}{6}$ possible committees. There are $\binom{5}{2}$, $\binom{3}{1}$, and $\binom{7}{3}$ ways to choose 2 Americans, 1 Canadian, and 3 Europeans, respectively. Therefore, by the generalized counting process, the probability is

$$\frac{\binom{5}{2}\binom{3}{1}\binom{7}{3}}{\binom{15}{6}} = \frac{30}{143} \approx 0.2097$$