Section 1.2 The Basic Principle of Counting

The Basic Principle of Counting: Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Proof: For $1 \leq i \leq m$ and $1 \leq j \leq n$ we say the outcome of the two experiments was (i, j) if the outcome of experiment 1 was i and experiment 2 was j. Now, simply write down all possible outcomes in the following way:

$$\begin{array}{llll} (1,1), & (1,2), & \ldots, & (1,n) \\ (2,1), & (2,2), & \ldots, & (2,n) \\ \ldots & & & \\ (m,1), & (m,2), & \ldots, & (m,n). \end{array}$$

Multiplying the number of rows (m) by the number of columns (n) shows there are a total number of mn possible outcomes. ■

EXAMPLES:

1. At a high school there are 12 teachers, with each teaching 4 courses. One teacher is to be given an award for best teaching *for a given course.* How many different choices of teacher/course are possible?

Solution: We regard the choice of teacher as the outcome of the first experiment. The outcome of the second experiment is one of the four courses that teacher is teaching. From the basic principle of counting we see there are $12 \times 4 = 48$ possible choices.

2. A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution: By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices.

The Generalized Basic Principle of Counting: If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if..., then there are a total of $n_1 n_2 \ldots n_r$ possible outcomes of the r experiments.

PROBLEMS:

1. Consider rolling 7 dice. How many possible outcomes are there?

2. There are 4 math, 3 chemistry, and 2 Spanish books on a shelf and you need one of each for the coming semester. How many ways can you choose one of each?

3. How many functions defined on n points are possible if each functional value is either 0 or 1?

4. Suppose a state's license plate consists of 3 numbers followed by 3 letters. However, no two letters or numbers can be the same (i.e. no repetition is allowed). How many different license plates can be made?

PROBLEMS:

1. Consider rolling 7 dice. How many possible outcomes are there?

Solution: For $i \leq 7$, let E_i be the possible outcomes of the ith die ($= \{1, 2, \ldots, 6\}$). Therefore, $n_i = 6$ for each i. By the above theorem, the total number of possibilities is

$$6 \times 6 \times \ldots \times 6 = 6^7.$$

2. There are 4 math, 3 chemistry, and 2 Spanish books on a shelf and you need one of each for the coming semester. How many ways can you choose one of each?

Solution: View the first experiment as choosing one of the math books, etc. There are

$$4 \times 3 \times 2 = 24$$

possibilities.

3. How many functions defined on n points are possible if each functional value is either 0 or 1?

Solution: Let the points be $1, 2, \ldots, n$. Since $f(i)$ must be either 0 or 1 for each $i = 1, 2, \ldots, n$, it follows that there are 2^n possible functions.

4. Suppose a state's license plate consists of 3 numbers followed by 3 letters. However, no two letters or numbers can be the same (i.e. no repetition is allowed). How many different license plates can be made?

Solution: The first number can be any of $\{0, 1, \ldots, 9\}$. So $n_1 = 10$. However, the second number can not be the first and so $n_2 = 9$. Similarly $n_3 = 8, n_4 = 26, n_5 = 25, n_6 = 24$. Thus, there are a total number of

$$10 \times 9 \times 8 \times 26 \times 25 \times 24 = 11,232,000.$$

Section 1.3 Permutations

DEFINITION: A *permutation* of a set X is a rearrangement of its elements.

EXAMPLES:

1. Let $X = \{1, 2\}$. Then there are 2 permutations:

$$12, \ 21$$

2. Let $X = \{1, 2, 3\}$. Then there are 6 permutations:

$$123, \ 132, \ 213, \ 231, \ 312, \ 321$$

3. Let $X = \{1, 2, 3, 4\}$. Then there are 24 permutations:

$$\begin{array}{cccccc} 1234, & 1243, & 1324, & 1342, & 1423, & 1432 \\ 2134, & 2143, & 2314, & 2341, & 2413, & 2431 \\ 3214, & 3241, & 3124, & 3142, & 3421, & 3412 \\ 4231, & 4213, & 4321, & 4312, & 4123, & 4132 \end{array}$$

REMARK: One can show that there are exactly $n!$ permutations of the n-element set X. In fact, we can use the generalized basic principle of counting. There are n possible outcomes of experiment 1, $n-1$ for the second, etc. Thus, there are

$$n! = n(n-1)(n-2)\ldots 2 \cdot 1$$

permutations of n objects.

EXAMPLES:

1. In a class with 8 students, how many ways can the students be lined up?

Solution: There are $8! = 40,320$ ways to line up the students.

2. A class consists of 15 students, 6 boys and 9 girls. Suppose there are also 3 instructors. Suppose that we want everyone to line up at random.

(a) How many ways can they line up?

(b) How many ways can they line up so that all the boys will be together, all the girls will be together, and all the instructors will be together?

Solution:

(a) There are a total of $18! \approx 6.4 \times 10^{15}$ ways for everyone to line up.

(b) There are $6!, 9!$, and $3!$ permutations of the boys, girls, and instructors respectively. Therefore, by the basic counting principle, there are $6!9!3!$ ways for them to line up with all boys, then girls, then instructors. But, that was one permutation of boys, girls, and instructors. There are $3!$ permutations of those sets (i.e. Boys/Girls/Inst., Girls/Boys/Inst., etc.). Each of those orderings also has $6!9!3!$ possible arrangements. Thus, the total number is $3!$ times $6!9!3!$, or $3!6!9!3! \approx 9.4 \times 10^9$.

3. How many different letter arrangements can be formed from the letters $PEPPER$?

Solution: First note there are a total of $6!$ different letter permutations. Pick one: $EPEPPR$. Note that we could permute the E's and not change the word, and we could permute the P's and not change the word. There are $2!$ ways to permute the E's and $3!$ ways to permute the P's. Multiplying (using the counting principle) shows that there are $3!2!$ permutations of the letters in $EPEPPR$ that do not change the word. Thus, this arrangement has been counted $3!2!$ times (in getting to the $6!$ number of total permutations), and so has every other word. Thus, there are

$$\frac{6!}{3!2!} = 60$$

unique letter arrangements.

4. Consider various ways of ordering the letters in the word $MISSISSIPPI$:

$$IIMSSPISSIP,\ ISSSPMIIPIS,\ PIMISSSSIIP, \quad \text{and so on.}$$

How many distinguishable orderings are there?

Solution 1: We first note that there are 11! permutations of the letters $MI_1S_1S_2I_2S_3S_4I_3P_1P_2I_4$ when the 4 I's, 4 S's and the 2 P's are distinguished from each other. However, consider any one of these permutations — for instance, $I_3P_2P_1I_4MI_2S_2S_1I_1S_3S_4$. If we now permute the I's among themselves, S's among themselves and P's among themselves, then the resultant arrangement would still be of the form $IPPIMISSISS$. That is, all 4!4!2! permutations are of the form $IPPIMISSISS$. Hence, there are

$$\frac{11!}{4!4!2!} = 34,650$$

possible letter arrangements of the letters $MISSISSIPPI$.

Solution 2: Since there are 11 positions in all, there are

$$\binom{11}{4} \text{ subsets of 4 positions for the } S\text{'s.}$$

Once the four S's are in place, there are

$$\binom{7}{4} \text{ subsets of 4 positions for the } I\text{'s.}$$

After the I's are in place, there are

$$\binom{3}{2} \text{ subsets of 2 positions for the } P\text{'s.}$$

That leaves just one position for the M. Hence, by the generalized basic principle of counting we have:

$$\left[\begin{array}{l}\text{number of ways to}\\ \text{position all the letters}\end{array}\right] = \binom{11}{4}\cdot\binom{7}{4}\cdot\binom{3}{2} = 34,650$$

THEOREM: In general, there are

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

EXAMPLE: How many different 10-letter codes can be made using three a's, four b's, and three c's?

Solution: By the Theorem above we have

$$\frac{10!}{3!4!3!} = 4,200$$

Section 1.4 Combinations

DEFINITION: We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.

EXAMPLES: We have

$$\binom{4}{2} = \frac{4!}{2! \cdot (4-2)!} = \frac{4!}{2! \cdot 2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2) \cdot (1 \cdot 2)} = \frac{3 \cdot 4}{1 \cdot 2} = 6$$

$$\binom{4}{3} = \frac{4!}{3! \cdot (4-3)!} = \frac{4!}{3! \cdot 1!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2 \cdot 3) \cdot 1} = \frac{4}{1} = 4$$

$$\binom{4}{4} = \frac{4!}{4! \cdot (4-4)!} = \frac{4!}{4! \cdot 0!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2 \cdot 3 \cdot 4) \cdot 1} = \frac{1}{1} = 1$$

$$\binom{8}{5} = \frac{8!}{5! \cdot (8-5)!} = \frac{8!}{5! \cdot 3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) \cdot (1 \cdot 2 \cdot 3)} = \frac{6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3} = \frac{7 \cdot 8}{1} = 56$$

EXAMPLES:

1. Suppose 5 members of a group of 12 are to be chosen to work as a team on a special project. How many distinct 5-person teams can be formed?

Solution: The number of distinct 5-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of 12. This number is

$$\binom{12}{5} = \frac{12!}{5! \cdot (12-5)!} = \frac{12!}{5! \cdot 7!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7)}$$

$$= \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 3 \cdot 5} = 3 \cdot 2 \cdot 11 \cdot 12 = 792$$

2. Suppose two members of the group of 12 refuse to work together on a team. How many distinct 5-person teams can be formed?

Solution: Call the two members of the group that refuse to work together A and B. We have:

$$\begin{aligned}\begin{bmatrix}\text{number of 5-person teams}\\ \text{that don't contain}\\ \text{both } A \text{ and } B\end{bmatrix} &= \begin{bmatrix}\text{total number of}\\ \text{5-person teams}\end{bmatrix} - \begin{bmatrix}\text{number of 5-person teams}\\ \text{that contain}\\ \text{both } A \text{ and } B\end{bmatrix}\\ &= \binom{12}{5} - \binom{10}{3} = 792 - 120 = 672\end{aligned}$$

3. Suppose the group of 12 consists of 5 men and 7 women. How many 5-person teams can be chosen that consist of 3 men and 2 women?

Solution: Note, that there are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven. Therefore, by the generalized basic principle of counting we have:

$$\begin{bmatrix}\text{number of 5-person teams that}\\ \text{contain 3 men and 2 women}\end{bmatrix} = \binom{5}{3} \cdot \binom{7}{2} = 210.$$

Properties of Binomial Coefficients

1. $\binom{n}{0} = \binom{n}{n} = 1$

Proof: We have

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = \frac{n!}{n! \cdot 1} = 1 \blacksquare$$

2. $\binom{n}{1} = \binom{n}{n-1} = n$

Proof: We have

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{(n-1)! \cdot n}{1! \cdot (n-1)!} = n$$

$$\binom{n}{n-1} = \frac{n!}{(n-1)![n-(n-1)]!} = \frac{n!}{(n-1)! \cdot 1!} = \frac{(n-1)! \cdot n}{(n-1)! \cdot 1!} = n \blacksquare$$

3. $\binom{n}{k} = \binom{n}{n-k}$

Proof: We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)![n-(n-k)]!} = \binom{n}{n-k} \quad \blacksquare$$

4. $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

Proof: We have

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{(k-1)!k(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k)!(n-k+1)} \\
&= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!} = \frac{n!(k+n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k} \quad \blacksquare
\end{aligned}$$

PROBLEM: Show that for all integers n and k with $1 \leq k \leq n$ we have

$$\binom{n}{k-1} + 2\binom{n}{k} + \binom{n}{k+1} = \binom{n+2}{k+1}$$

Proof: By property 4 we have

$$\binom{n}{k-1} + 2\binom{n}{k} + \binom{n}{k+1} = \binom{n}{k-1} + \binom{n}{k} + \binom{n}{k} + \binom{n}{k+1}$$

$$= \binom{n+1}{k} + \binom{n+1}{k+1} = \binom{n+2}{k+1} \blacksquare$$

THEOREM (The Binomial Theorem): Let x and y be any real numbers and let n be any nonnegative integer. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad (*)$$

or

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Proof of the Binomial Theorem by Induction: When $n = 1$, equation $(*)$ reduces to

$$x + y = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0 = y + x$$

Assume equation $(*)$ for $n-1$. Now,

$$\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}
\end{aligned}$$

Letting $i = k+1$ in the first sum and $i = k$ in the second sum, we find that

$$\begin{aligned}
(x+y)^n &= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right] x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-i} \binom{n}{i} x^i y^{n-i} + y^n \\
&= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}
\end{aligned}$$

where the next-to-last equality follows by property 4. By induction the theorem is now proved. $\blacksquare$

Combinatorial Proof of the Binomial Theorem: Consider the product

$$(x_1 + y_1)(x_2 + y_2) \dots (x_n + y_n)$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either x_i or y_i for each. $i = 1, 2, \dots, n$. For example,

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$$

Now, how many of the 2^n terms in the sum will have k of the x_i's and $(n-k)$ of the y_i's? As each term consisting of k of the x_i's and $(n-k)$ of the y_i's conesponds to a choice of a group of k from the n values $x_1, x_2, \dots, x_n$, there are $\binom{n}{k}$ such terms. Thus, letting $x_i = x, y_i = y, i = 1, \dots, n$, we see that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \ \blacksquare$$

PROBLEM: For all integers $n \geq 1$ we have

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n.$$

Proof: Putting $x = y = 1$ in the Theorem above, we get

$$(1+1)^n = \binom{n}{0} \cdot 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot 1 + \binom{n}{2} \cdot 1^{n-2} \cdot 1^2 + \ldots + \binom{n}{n-1} \cdot 1 \cdot 1^{n-1} + \binom{n}{n} \cdot 1^n,$$

hence

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n}. \ \blacksquare$$

PROBLEM: For all integers $n \geq 1$ we have

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0.$$

Proof: Putting $x = 1$ and $y = -1$ in the Theorem above, we get

$$(1-1)^n$$

$$= \binom{n}{0} \cdot 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot (-1) + \binom{n}{2} \cdot 1^{n-2} \cdot (-1)^2 + \ldots + \binom{n}{n-1} \cdot 1 \cdot (-1)^{n-1} + \binom{n}{n} \cdot (-1)^n,$$

hence

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n}. \ \blacksquare$$

PROBLEM: Are there integer numbers n, k such that

$$\binom{n}{k} = 1001, \quad \binom{n}{k+1} = 2002, \quad \binom{n}{k+2} = 3003?$$

$$(x+1)^{0}=1$$

$$(x+1)^{1}=x+1$$

$$(x+1)^{2}=x^{2}+2x+1$$

$$(x+1)^{3}=x^{3}+3x^{2}+3x+1$$

$$(x+1)^{4}=x^{4}+4x^{3}+6x^{2}+4x+1$$

$$(x+1)^{5}=x^{5}+5x^{4}+10x^{3}+10x^{2}+5x+1$$

$$(x+1)^{6}=x^{6}+6x^{5}+15x^{4}+20x^{3}+15x^{2}+6x+1$$

$$(x+1)^{7}=x^{7}+7x^{6}+21x^{5}+35x^{4}+35x^{3}+21x^{2}+7x+1$$

$$(x+1)^{8}=x^{8}+8x^{7}+28x^{6}+56x^{5}+70x^{4}+56x^{3}+28x^{2}+8x+1$$

$$(x+1)^{9}=x^{9}+9x^{8}+36x^{7}+84x^{6}+126x^{5}+126x^{4}+84x^{3}+36x^{2}+9x+1$$

$$(x+1)^{10}=x^{10}+10x^{9}+45x^{8}+120x^{7}+210x^{6}+252x^{5}+210x^{4}+120x^{3}+45x^{2}+10x+1$$

$$(x+1)^{11}=x^{11}+11x^{10}+55x^{9}+165x^{8}+330x^{7}+462x^{6}+462x^{5}+330x^{4}+165x^{3}+55x^{2}+11x+1$$

$$(x+1)^{12}=x^{12}+12x^{11}+66x^{10}+220x^{9}+495x^{8}+792x^{7}+924x^{6}+792x^{5}+495x^{4}+220x^{3}+66x^{2}+12x+1$$

$$(x+1)^{13}=x^{13}+13x^{12}+78x^{11}+286x^{10}+715x^{9}+1287x^{8}+1716x^{7}+1716x^{6}+1287x^{5}+715x^{4}+286x^{3}+78x^{2}+13x+1$$

$$(x+1)^{14}=x^{14}+14x^{13}+91x^{12}+364x^{11}+1001x^{10}+2002x^{9}+3003x^{8}+3432x^{7}+3003x^{6}+2002x^{5}+1001x^{4}+364x^{3}+91x^{2}+14x+1$$

$$(x+1)^{15}=x^{15}+15x^{14}+105x^{13}+455x^{12}+1365x^{11}+3003x^{10}+5005x^{9}+6435x^{8}+6435x^{7}+5005x^{6}+3003x^{5}+1365x^{4}+455x^{3}+105x^{2}+15x+1$$

$$(x+1)^{16}=x^{16}+16x^{15}+120x^{14}+560x^{13}+1820x^{12}+4368x^{11}+8008x^{10}+11440x^{9}+12870x^{8}+11440x^{7}+8008x^{6}+4368x^{5}+1820x^{4}+560x^{3}+120x^{2}+16x+1$$

$$(x+1)^{17}=x^{17}+17x^{16}+136x^{15}+680x^{14}+2380x^{13}+6188x^{12}+12376x^{11}+19448x^{10}+24310x^{9}+24310x^{8}+19448x^{7}+12376x^{6}+6188x^{5}+2380x^{4}+680x^{3}+136x^{2}+17x+1$$

$$(x+1)^{18}=x^{18}+18x^{17}+153x^{16}+816x^{15}+3060x^{14}+8568x^{13}+18564x^{12}+31824x^{11}+43758x^{10}+48620x^{9}+43758x^{8}+31824x^{7}+18564x^{6}+8568x^{5}+3060x^{4}+816x^{3}+153x^{2}+18x+1$$

Section 1.5 Multinomial Coefficients

Suppose that you have n objects and want to separate them into r distinct groups of size $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$.

Question: How many ways can this be done?

Solution: Note that there are $\binom{n}{n_1}$ possible choices for the first group. For each such choice, there are $\binom{n-n_1}{n_2}$ possible choices for the second group. Given a choice of the first and second group, there are $\binom{n-n_1-n_2}{n_3}$ choices for the third group, etc. Therefore, by the generalized counting principle there are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots\binom{n-n_1-\ldots-n_{r-1}}{n_r}$$

$$= \frac{n!}{(n-n_1)!n_1!} \cdot \frac{(n-n_1)!}{(n-n_1-n_2)!n_2!} \cdot \frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!n_3!} \cdot \ldots \frac{(n-n_1-\cdots-n_{r-1})!}{0!n_r!}$$

$$= \frac{n!}{n_1!n_2!\ldots n_r!}$$

possible divisions.

NOTATION: If $n_1 + n_2 + \ldots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}$$

EXAMPLE: There are 9 students in a class. For a project, 4 will do field research, 3 will write the project up, and 2 will present the work. How many ways can the students be divided into the different groups?

Solution: Direct application: $\frac{9!}{4!3!2!} = 1,260$ ways.

REMARK: Note that order was important above.

EXAMPLE: A group of 8 students is to be divided into two groups of four. How many ways can this happen?

Solution: There are $8!/(4!4!) = 70$ ways to divide the students into a *first* group and a *second* group. But this ordering is arbitrary and so each partition has been double counted. Thus, the correct number is $8!/(4!4! \times 2) = 35$.

THEOREM (The Multinomial Theorem):

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{\substack{(n_1, \ldots, n_r) : \\ n_1 + \ldots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \ldots + n_r = n$.