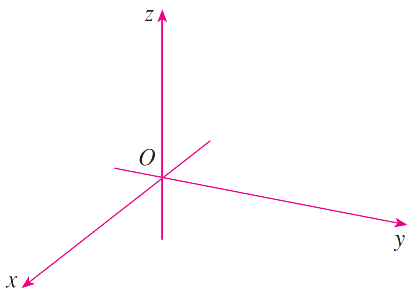
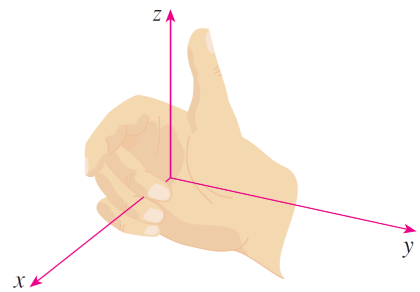
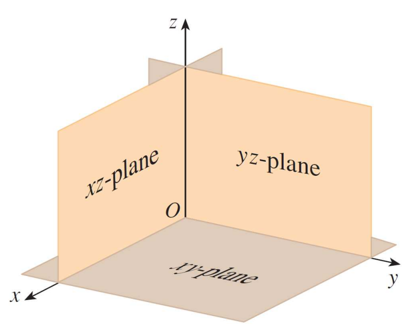
Section 9.3 Three-Dimensional Coordinate Geometry

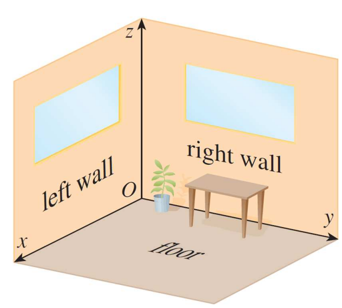
In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in the first Figure below. The direction of the z-axis is determined by the **right-hand rule** as illustrated in the second Figure below: If you curl the fingers of your right hand around the z-axis in the direction of a $90°$ counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.



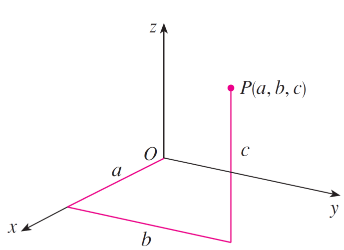


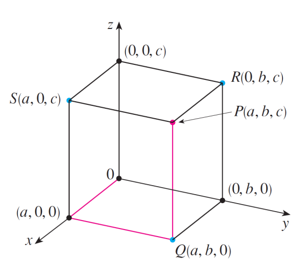
The three coordinate axes determine the three **coordinate planes** illustrated in first Figure below. The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.

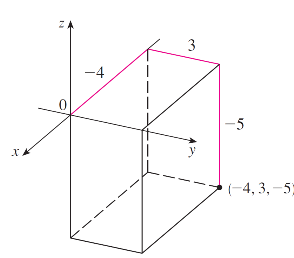


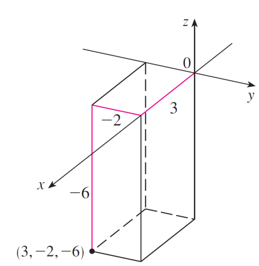


Now if P is any point in space, let a be the (directed) distance from the yz-plane to P, let b be the distance from the xz-plane to P and let c be the distance from the xy-plane to P. We represent the point P by the **ordered triple** (a, b, c). The point $P(a, b, c)$ determines a rectangular box as in the second Figure below. As numerical illustrations, the points $(-4, 3, -5)$ and $(3, -2, -6)$ are plotted in the last two Figures below. The set of all ordered triples $\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ forms a **three-dimensional rectangular coordinate system**.







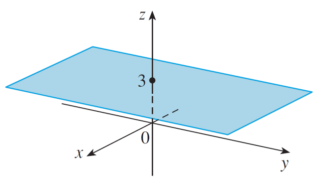


In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

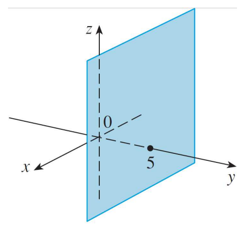
EXAMPLE: What surfaces or curves are represented by the following equations?

(a) $z = 3$ in $\mathbb{R}^3$ (b) $y = 5$ in $\mathbb{R}^3$ (c) $y = 5$ in $\mathbb{R}^2$ (d) $y = x$ in $\mathbb{R}^3$

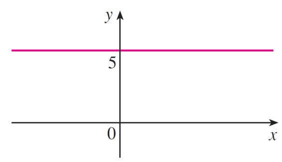
Solution:



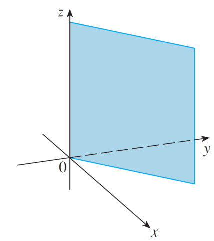
(a) $z = 3$, a plane in $\mathbb{R}^3$

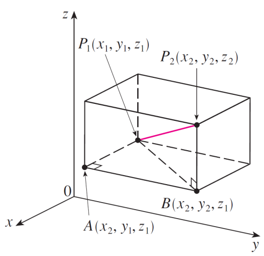


(b) $y = 5$, a plane in $\mathbb{R}^3$



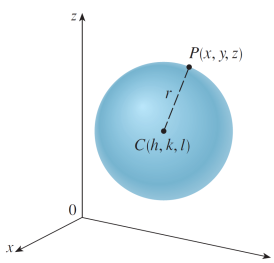
(c) $y = 5$, a line in $\mathbb{R}^2$





Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$