Section 9.2 The Dot Product

The Dot Product of Vectors

Definition of the Dot Product

If $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$ are vectors, then their **dot product,** denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = a_1 a_2 + b_1 b_2$$

EXAMPLES:

(a) If $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 5 \rangle$ then

$$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(5) = 12 - 10 = 2$$

(b) If $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 6\mathbf{j}$ then

$$\mathbf{u} \cdot \mathbf{v} = (2)(5) + (1)(-6) = 10 - 6 = 4$$

Properties of the Dot Product

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$

Proof: We prove only the last property. Let $\mathbf{u} = \langle a, b \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = \langle a, b \rangle \cdot \langle a, b \rangle = a \cdot a + b \cdot b = a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |\mathbf{u}|^2$$

EXAMPLE: If $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 5 \rangle$ then

$$|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} = (3)(3) + (-2)(-2) = 9 + 4 = 13$$

and

$$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = (4)(4) + (5)(5) = 16 + 25 = 41$$

therefore

$$|\mathbf{u}| = \sqrt{13} \quad \text{and} \quad |\mathbf{v}| = \sqrt{41}$$

The Dot Product Theorem

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

Angle between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

EXAMPLE: Find the angle between the vectors $\mathbf{u} = \langle 2, 5 \rangle$ and $\mathbf{v} = \langle 4, -3 \rangle$.

Solution: By the formula for the angle between two vectors, we have

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(2)(4) + (5)(-3)}{\sqrt{2^2 + 5^2}\sqrt{4^2 + 3^2}} = \frac{8 - 15}{\sqrt{4 + 25}\sqrt{16 + 9}} = \frac{-7}{\sqrt{29}\sqrt{25}} = -\frac{7}{5\sqrt{29}}$$

Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\theta = \cos^{-1}\left(-\frac{7}{5\sqrt{29}}\right) \approx 105.1^\circ$$

Orthogonal Vectors

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

EXAMPLE: Determine whether the vectors in each pair are perpendicular.

(a) $\mathbf{u} = \langle 3, 5 \rangle$ and $\mathbf{v} = \langle 2, -8 \rangle$

(b) $\mathbf{u} = \langle 2, 1 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$

Solution:

(a) We have

$$\mathbf{u} \cdot \mathbf{v} = (3)(2) + (5)(-8) = 6 - 40 = -34 \neq 0$$

so $\mathbf{u}$ and $\mathbf{v}$ are not perpendicular.

(b) We have

$$\mathbf{u} \cdot \mathbf{v} = (2)(-1) + (1)(2) = -2 + 2 = 0$$

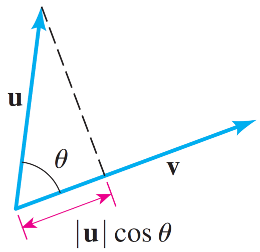
so $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

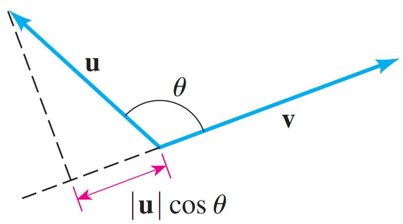
The Component of u Along v

The **component of u along v** (or the **component of u in the direction of v**) is defined to be

$$|\mathbf{u}|\cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.





Calculating Components

The component of $\mathbf{u}$ along $\mathbf{v}$ is $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$.

Proof: Since $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$, we have

$$\text{component of } \mathbf{u} \text{ along } \mathbf{v} \ = |\mathbf{u}|\cos\theta = |\mathbf{u}|\cdot\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

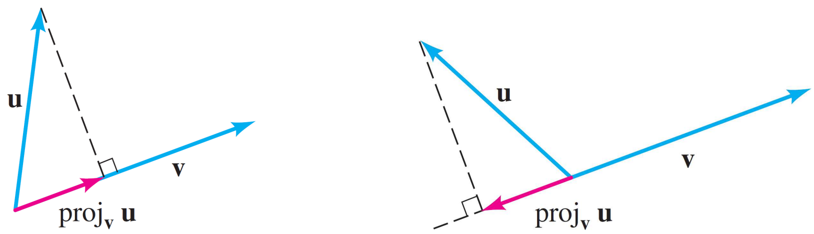
EXAMPLE: Let $\mathbf{u} = \langle 1, 4\rangle$ and $\mathbf{v} = \langle -2, 1\rangle$. Find the component of $\mathbf{u}$ along $\mathbf{v}$.

Solution: We have

$$\text{component of } \mathbf{u} \text{ along } \mathbf{v} \ = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{(1)(-2)+(4)(1)}{\sqrt{4+1}} = \frac{2}{\sqrt{5}}$$

The Projection of u onto v

The projection of $\mathbf{u}$ onto $\mathbf{v}$, denoted by $\text{proj}_{\mathbf{v}}\mathbf{u}$, is the vector whose *direction* is the same as $\mathbf{v}$ and whose *length* is the component of $\mathbf{u}$ along $\mathbf{v}$.



To find an expression for $\text{proj}_{\mathbf{v}}\mathbf{u}$, we first find a unit vector in the direction of $\mathbf{v}$ and then multiply it by the component of $\mathbf{u}$ along $\mathbf{v}$:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = (\text{component of } \mathbf{u} \text{ along } \mathbf{v})(\text{unit vector in direction of } \mathbf{v}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

Calculating Projections

The **projection of u onto v** is the vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ given by

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

If the vector $\mathbf{u}$ is **resolved** into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$, then

$$\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u} \qquad \text{and} \qquad \mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$$

EXAMPLE: Let $\mathbf{u} = \langle -2, 9\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$.

(a) Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

(b) Resolve $\mathbf{u}$ into $\mathbf{u_1}$ and $\mathbf{u_2}$, where $\mathbf{u_1}$ is parallel to $\mathbf{v}$ and $\mathbf{u_2}$ is orthogonal to $\mathbf{v}$.

Solution:

(a) By the formula for the projection of one vector onto another we have

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \left(\frac{\langle -2, 9\rangle\cdot\langle -1, 2\rangle}{(-1)^2+2^2}\right)\langle -1, 2\rangle = \left(\frac{(-2)(-1)+(9)(2)}{1+4}\right)\langle -1, 2\rangle$$
$$= \left(\frac{2+18}{5}\right)\langle -1, 2\rangle = 4\langle -1, 2\rangle = \langle -4, 8\rangle$$

(b) By the formula in the preceding box we have $\mathbf{u} = \mathbf{u_1} + \mathbf{u_2}$, where

$$\mathbf{u_1} = \text{proj}_{\mathbf{v}}\mathbf{u} = \langle -4, 8\rangle$$
$$\mathbf{u_2} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle -2, 9\rangle - \langle -4, 8\rangle = \langle 2, 1\rangle$$