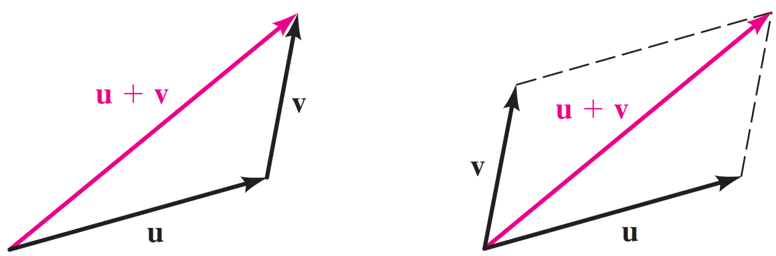
Section 9.1 Vectors in Two Dimensions

Geometric Description of Vectors

A **vector** in the plane is a line segment with an assigned direction. We sketch a vector as shown in the first Figure below with an arrow to specify the direction. We denote this vector by $\overrightarrow{AB}$. Point A is the **initial point**, and B is the **terminal point** of the vector $\overrightarrow{AB}$. The length of the line segment AB is called the **magnitude** or **length** of the vector and is denoted by $|\overrightarrow{AB}|$. We use boldface letters to denote vectors. Thus, we write $\mathbf{u} = \overrightarrow{AB}$. Two vectors are considered **equal** if they have equal magnitude and the same direction.

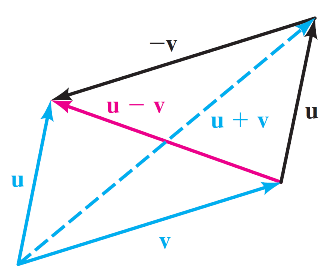
The Figures below illustrate the way we

(a) add

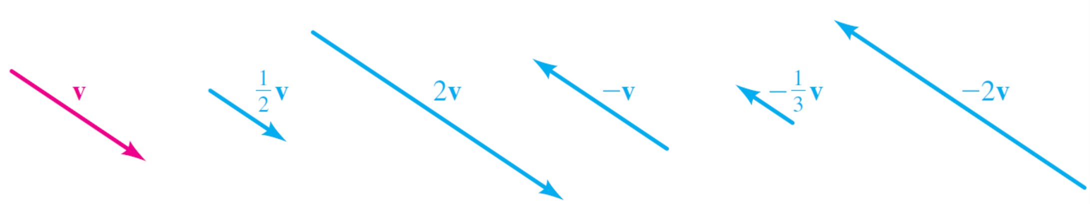


(a) (b)

(b) subtract



(c) multiply a vector by a scalar

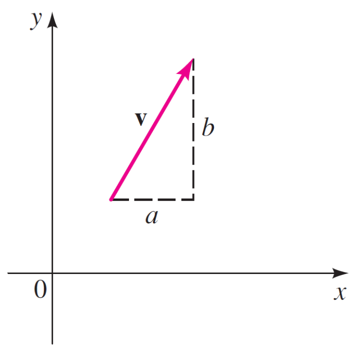


Vectors in the Coordinate Plane

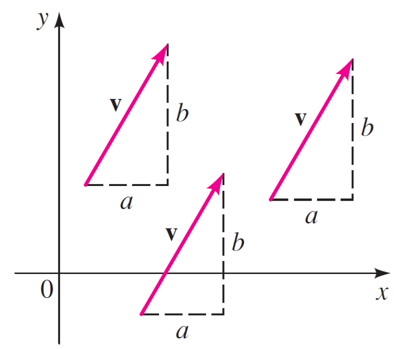
By placing a vector in a coordinate plane, we can describe it analytically (that is, by using components). We represent a vector $\mathbf{v}$ as an ordered pair of real numbers.

$$\mathbf{v} = \langle a, b \rangle$$

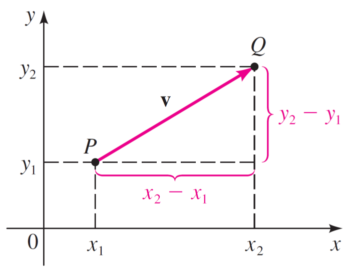
where a is the **horizontal component** of $\mathbf{v}$ and b is the **vertical component** of $\mathbf{v}$. The vector $\langle a, b \rangle$ has many different representations, depending on its initial point.



(a)



(b)



Component Form of a Vector

If a vector $\mathbf{v}$ is represented in the plane with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$, then

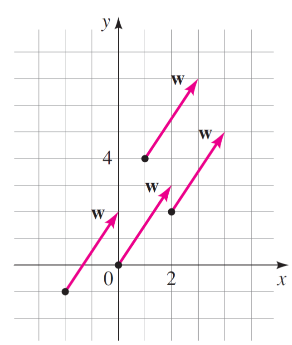
$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

EXAMPLE:

(a) Find the component form of the vector $\mathbf{u}$ with initial point $(-2, 5)$ and terminal point $(3, 7)$.

(b) If the vector $\mathbf{v} = \langle 3, 7 \rangle$ is sketched with initial point $(2, 4)$, what is its terminal point?

(c) Sketch representations of the vector $\mathbf{w} = \langle 2, 3 \rangle$ with initial points at $(0, 0)$, $(2, 2)$, $(-2, -1)$, and $(1, 4)$.



Solution:

(a) The desired vector is

$$\mathbf{u} = \langle 3 - (-2), 7 - 5 \rangle = \langle 5, 2 \rangle$$

(b) Let the terminal point of $\mathbf{v}$ be (x, y). Then

$$\langle x - 2, y - 4 \rangle = \langle 3, 7 \rangle$$

So $x - 2 = 3$ and $y - 4 = 7$, or $x = 5$ and $y = 11$. The terminal point is $(5, 11)$.

(c) Representations of the vector $\mathbf{w}$ are sketched in the Figure above.

EXAMPLE:

(a) Find the vector $\mathbf{u}$ with initial point $(-5, 4)$ and terminal point $(4, 8)$.

(b) If the vector $\mathbf{v} = \langle 4, 7 \rangle$ is sketched with initial point $(4, 5)$, what is its terminal point?

EXAMPLE:

(a) Find the vector $\mathbf{u}$ with initial point $(-5, 4)$ and terminal point $(4, 8)$.

(b) If the vector $\mathbf{v} = \langle 4, 7\rangle$ is sketched with initial point $(4, 5)$, what is its terminal point?

Solution:

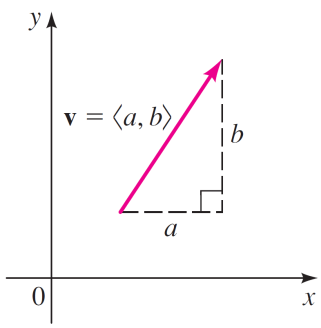
(a) The desired vector is

$$\mathbf{u} = \langle 4 - (-5), 8 - 4\rangle = \langle 9, 4\rangle$$

(b) Let the terminal point of $\mathbf{v}$ be (x, y). Then

$$\langle x - 4, y - 5\rangle = \langle 4, 7\rangle$$

So $x - 4 = 4$ and $y - 5 = 7$, or $x = 8$ and $y = 12$. The terminal point is $(8, 12)$.



Magnitude of a Vector

The **magnitude** or **length** of a vector $\mathbf{v} = \langle a, b\rangle$ is

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

EXAMPLE: Find the magnitude of each vector.

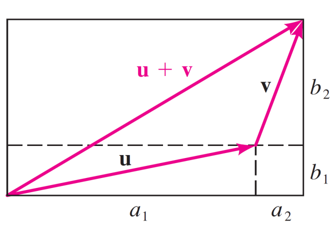
(a) $\mathbf{u} = \langle 2, -3\rangle$ (b) $\mathbf{v} = \langle 5, 0\rangle$ (c) $\mathbf{w} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$

Solution:

(a) $|\mathbf{u}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

(b) $|\mathbf{v}| = \sqrt{5^2 + 0^2} = \sqrt{25} = 5$

(c) $|\mathbf{w}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{9 + 16}{25}} = 1$



Algebraic Operations on Vectors

If $\mathbf{u} = \langle a_1, b_1\rangle$ and $\mathbf{v} = \langle a_2, b_2\rangle$, then

$$\mathbf{u} + \mathbf{v} = \langle a_1 + a_2, b_1 + b_2\rangle$$

$$\mathbf{u} - \mathbf{v} = \langle a_1 - a_2, b_1 - b_2\rangle$$

$$c\mathbf{u} = \langle ca_1, cb_1\rangle, \qquad c \in \mathbb{R}$$

EXAMPLE: If $\mathbf{u} = \langle 2, -3\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, $-3\mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v}$.

EXAMPLE: If $\mathbf{u} = \langle 2, -3\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$, find $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $2\mathbf{u}$, $-3\mathbf{v}$, and $2\mathbf{u}+3\mathbf{v}$.

Solution: We have

$$\mathbf{u}+\mathbf{v} = \langle 2,-3\rangle + \langle -1,2\rangle = \langle 2+(-1), -3+2\rangle = \langle 1,-1\rangle$$

$$\mathbf{u}-\mathbf{v} = \langle 2,-3\rangle - \langle -1,2\rangle = \langle 2-(-1), -3-2\rangle = \langle 3,-5\rangle$$

$$2\mathbf{u} = 2\langle 2,-3\rangle = \langle 2(2), 2(-3)\rangle = \langle 4,-6\rangle$$

$$-3\mathbf{v} = -3\langle -1,2\rangle = \langle (-3)(-1), (-3)2\rangle = \langle 3,-6\rangle$$

$$2\mathbf{u}+3\mathbf{v} = 2\langle 2,-3\rangle + 3\langle -1,2\rangle = \langle 4,-6\rangle + \langle -3,6\rangle = \langle 4+(-3), -6+6\rangle = \langle 1,0\rangle$$

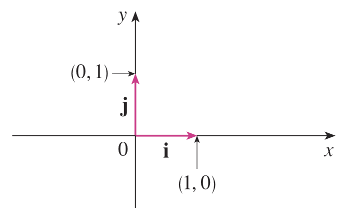
Properties of Vectors

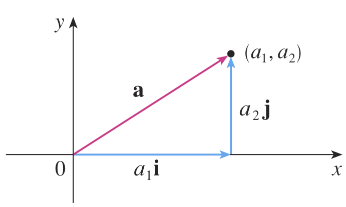
Vector addition	Multiplication by a scalar
$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$	$c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$
$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$	$(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$
$\mathbf{u}+\mathbf{0} = \mathbf{u}$	$(cd)\mathbf{u} = c(d\mathbf{u}) = d(c\mathbf{u})$
$\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$	$1\mathbf{u} = \mathbf{u}$
Length of a vector	$0\mathbf{u} = \mathbf{0}$
$\lvert c\mathbf{u}\rvert = \lvert c\rvert\,\lvert\mathbf{u}\rvert$	$c\mathbf{0} = \mathbf{0}$

The **zero vector** is the vector $\mathbf{0} = \langle 0,0\rangle$. A vector of length 1 is called a **unit vector**. For instance, in the Example above, the vector $\mathbf{w} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$ is a unit vector. Two useful unit vectors are $\mathbf{i}$ and $\mathbf{j}$, defined by

$$\mathbf{i} = \langle 1,0\rangle \qquad \mathbf{j} = \langle 0,1\rangle$$

These vectors are special because any vector can be expressed in terms of them.





Vectors in Terms of i and j

The vector $\mathbf{v} = \langle a, b\rangle$ can be expressed in terms of $\mathbf{i}$ and $\mathbf{j}$ by

$$\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$$

EXAMPLE:

(a) Write the vector $\mathbf{u} = \langle 5, -8\rangle$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) If $\mathbf{u} = 3\mathbf{i}+2\mathbf{j}$ and $\mathbf{v} = -\mathbf{i}+6\mathbf{j}$, write $2\mathbf{u}+5\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

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(a) Write the vector $\mathbf{u} = \langle 5, -8 \rangle$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

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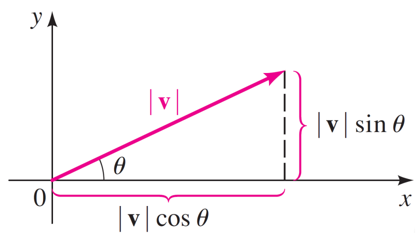
Solution:

(a) $\mathbf{u} = 5\mathbf{i} + (-8)\mathbf{j} = 5\mathbf{i} - 8\mathbf{j}$

(b) The properties of addition and scalar multiplication of vectors show that we can manipulate vectors in the same way as algebraic expressions. Thus

$$\begin{aligned} 2\mathbf{u} + 5\mathbf{v} &= 2(3\mathbf{i} + 2\mathbf{j}) + 5(-\mathbf{i} + 6\mathbf{j}) \\ &= (6\mathbf{i} + 4\mathbf{j}) + (-5\mathbf{i} + 30\mathbf{j}) \\ &= (6 - 5)\mathbf{i} + (4 + 30)\mathbf{j} \\ &= \mathbf{i} + 34\mathbf{j} \end{aligned}$$



Horizontal and Vertical Components of a Vector

Let $\mathbf{v}$ be a vector with magnitude $|\mathbf{v}|$ and direction θ. Then $\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$, where

$$a = |\mathbf{v}| \cos\theta \qquad \text{and} \qquad b = |\mathbf{v}| \sin\theta$$

Thus, we can express $\mathbf{v}$ as

$$\mathbf{v} = |\mathbf{v}| \cos\theta\, \mathbf{i} + |\mathbf{v}| \sin\theta\, \mathbf{j}$$

EXAMPLE:

(a) A vector $\mathbf{v}$ has length 8 and direction $\pi/3$. Find the horizontal and vertical components, and write $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) Find the direction of the vector $\mathbf{u} = -\sqrt{3}\mathbf{i} + \mathbf{j}$.

Solution:

(a) We have $\mathbf{v} = \langle a, b \rangle$, where the components are given by

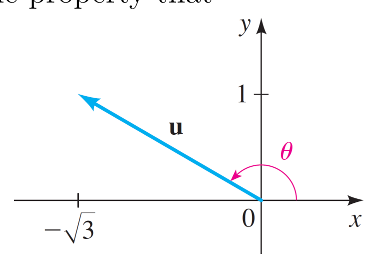
$$a = 8\cos\frac{\pi}{3} = 8 \cdot \frac{1}{2} = 4 \qquad \text{and} \qquad b = 8\sin\frac{\pi}{3} = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$$

Thus, $\mathbf{v} = \left\langle 4, 4\sqrt{3} \right\rangle = 4\mathbf{i} + 4\sqrt{3}\mathbf{j}$.

(b) From the Figure on the right we see that the direction θ has the property that

$$\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Thus, the reference angle for θ is $\pi/6$. Since the terminal point of the vector $\mathbf{u}$ is in Quadrant II, it follows that $\theta = 5\pi/6$.



EXAMPLE: A woman launches a boat from one shore of a straight river and wants to land at the point directly on the opposite shore. If the speed of the boat (relative to the water) is 10 mi/h and the river is flowing east at the rate of 5 mi/h, in what direction should she head the boat in order to arrive at the desired landing point?

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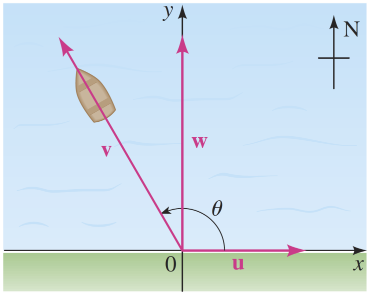

Solution: We choose a coordinate system with the origin at the initial position of the boat as shown in the Figure above. Let $\mathbf{u}$ and $\mathbf{v}$ represent the velocities of the river and the boat, respectively. Clearly, $\mathbf{u} = 5\mathbf{i}$, and since the speed of the boat is 10 mi/h, we have $|\mathbf{v}| = 10$, so

$$\mathbf{v} = (10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j}$$

where the angle θ is as shown in the Figure above. The true course of the boat is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$. We have

$$\begin{aligned}\mathbf{w} = \mathbf{u} + \mathbf{v} &= 5\mathbf{i} + (10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j} \\ &= (5 + 10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j}\end{aligned}$$

Since the woman wants to land at a point directly across the river, her direction should have horizontal component 0. In other words, she should choose θ in such a way that

$$\begin{aligned}5 + 10\cos\theta &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= 120^\circ\end{aligned}$$

Thus she should head the boat in the direction $\theta = 120^\circ$ (or N 30° W).