Section 9.1 Vectors in Two Dimensions

Geometric Description of Vectors

A vector in the plane is a line segment with an assigned direction. We sketch a vector as shown in the first Figure below with an arrow to specify the direction. We denote this vector by $\overrightarrow{AB}$. Point $A$ is the initial point, and $B$ is the terminal point of the vector $\overrightarrow{AB}$. The length of the line segment $AB$ is called the magnitude or length of the vector and is denoted by $|\overrightarrow{AB}|$. We use boldface letters to denote vectors. Thus, we write $\mathbf{u} = \overrightarrow{AB}$. Two vectors are considered equal if they have equal magnitude and the same direction.

The Figures below illustrate the way we

(a) add

(b) subtract

(c) multiply a vector by a scalar
Vectors in the Coordinate Plane

By placing a vector in a coordinate plane, we can describe it analytically (that is, by using components). We represent a vector \( \mathbf{v} \) as an ordered pair of real numbers.

\[ \mathbf{v} = \langle a, b \rangle \]

where \( a \) is the horizontal component of \( \mathbf{v} \) and \( b \) is the vertical component of \( \mathbf{v} \). The vector \( \langle a, b \rangle \) has many different representations, depending on its initial point.

**EXAMPLE:**

(a) Find the component form of the vector \( \mathbf{u} \) with initial point \((-2, 5)\) and terminal point \((3, 7)\).

(b) If the vector \( \mathbf{v} = \langle 3, 7 \rangle \) is sketched with initial point \((2, 4)\), what is its terminal point?

(c) Sketch representations of the vector \( \mathbf{w} = \langle 2, 3 \rangle \) with initial points at \((0, 0)\), \((2, 2)\), \((-2, -1)\), and \((1, 4)\).

**Solution:**

(a) The desired vector is

\[ \mathbf{u} = \langle 3 - (-2), 7 - 5 \rangle = \langle 5, 2 \rangle \]

(b) Let the terminal point of \( \mathbf{v} \) be \((x, y)\). Then

\[ \langle x - 2, y - 4 \rangle = \langle 3, 7 \rangle \]

So \( x - 2 = 3 \) and \( y - 4 = 7 \), or \( x = 5 \) and \( y = 11 \). The terminal point is \((5, 11)\).

(c) Representations of the vector \( \mathbf{w} \) are sketched in the Figure above.

**EXAMPLE:**

(a) Find the vector \( \mathbf{u} \) with initial point \((-5, 4)\) and terminal point \((4, 8)\).

(b) If the vector \( \mathbf{v} = \langle 4, 7 \rangle \) is sketched with initial point \((4, 5)\), what is its terminal point?
EXAMPLE:

(a) Find the vector \( \mathbf{u} \) with initial point \((-5, 4)\) and terminal point \((4, 8)\).

(b) If the vector \( \mathbf{v} = \langle 4, 7 \rangle \) is sketched with initial point \((4, 5)\), what is its terminal point?

Solution:

(a) The desired vector is
\[
\mathbf{u} = (4 - (-5), 8 - 4) = \langle 9, 4 \rangle
\]

(b) Let the terminal point of \( \mathbf{v} \) be \((x, y)\). Then
\[
\langle x - 4, y - 5 \rangle = \langle 4, 7 \rangle
\]

So \(x - 4 = 4\) and \(y - 5 = 7\), or \(x = 8\) and \(y = 12\). The terminal point is \((8, 12)\).

EXAMPLE: Find the magnitude of each vector.

(a) \( \mathbf{u} = \langle 2, -3 \rangle \) 

(b) \( \mathbf{v} = \langle 5, 0 \rangle \) 

(c) \( \mathbf{w} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \)

Solution:

(a) \(|\mathbf{u}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}\)

(b) \(|\mathbf{v}| = \sqrt{5^2 + 0^2} = \sqrt{25} = 5\)

(c) \(|\mathbf{w}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1\)

EXAMPLE: If \( \mathbf{u} = \langle 2, -3 \rangle \) and \( \mathbf{v} = \langle -1, 2 \rangle \), find \( \mathbf{u} + \mathbf{v} \), \( \mathbf{u} - \mathbf{v} \), \( 2\mathbf{u} \), \(-3\mathbf{v} \), and \(2\mathbf{u} + 3\mathbf{v}\).
EXAMPLE: If \( \mathbf{u} = \langle 2, -3 \rangle \) and \( \mathbf{v} = \langle -1, 2 \rangle \), find \( \mathbf{u} + \mathbf{v} \), \( \mathbf{u} - \mathbf{v} \), \( 2\mathbf{u} \), \(-3\mathbf{v} \), and \( 2\mathbf{u} + 3\mathbf{v} \).

Solution: We have

\[
\begin{align*}
\mathbf{u} + \mathbf{v} &= \langle 2, -3 \rangle + \langle -1, 2 \rangle = \langle 2 + (-1), -3 + 2 \rangle = \langle 1, -1 \rangle \\
\mathbf{u} - \mathbf{v} &= \langle 2, -3 \rangle - \langle -1, 2 \rangle = \langle 2 - (-1), -3 - 2 \rangle = \langle 3, -5 \rangle \\
2\mathbf{u} &= 2 \langle 2, -3 \rangle = \langle 2(2), 2(-3) \rangle = \langle 4, -6 \rangle \\
-3\mathbf{v} &= -3 \langle -1, 2 \rangle = \langle (-3)(-1), (-3)2 \rangle = \langle 3, -6 \rangle \\
2\mathbf{u} + 3\mathbf{v} &= 2 \langle 2, -3 \rangle + 3 \langle -1, 2 \rangle = \langle 4, -6 \rangle + \langle -3, 6 \rangle = \langle 4 + (-3), -6 + 6 \rangle = \langle 1, 0 \rangle
\end{align*}
\]

**Properties of Vectors**

<table>
<thead>
<tr>
<th>Vector addition</th>
<th>Multiplication by a scalar</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} )</td>
<td>( c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} )</td>
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<tr>
<td>( \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} )</td>
<td>( (c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u} )</td>
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<tr>
<td>( \mathbf{u} + \mathbf{0} = \mathbf{u} )</td>
<td>( (cd)\mathbf{u} = c(d\mathbf{u}) = d(c\mathbf{u}) )</td>
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<tr>
<td>( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} )</td>
<td>( 1\mathbf{u} = \mathbf{u} )</td>
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<tr>
<td>Length of a vector</td>
<td>( 0\mathbf{u} = \mathbf{0} )</td>
</tr>
<tr>
<td>(</td>
<td>c\mathbf{u}</td>
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</tbody>
</table>

The **zero vector** is the vector \( \mathbf{0} = \langle 0, 0 \rangle \). A vector of length 1 is called a **unit vector**. For instance, in the Example above, the vector \( \mathbf{w} = \langle \frac{3}{5}, \frac{4}{5} \rangle \) is a unit vector. Two useful unit vectors are \( \mathbf{i} \) and \( \mathbf{j} \), defined by

\[
\mathbf{i} = \langle 1, 0 \rangle \quad \mathbf{j} = \langle 0, 1 \rangle
\]

These vectors are special because any vector can be expressed in terms of them.

**Vectors in Terms of \( \mathbf{i} \) and \( \mathbf{j} \)**

The vector \( \mathbf{v} = \langle a, b \rangle \) can be expressed in terms of \( \mathbf{i} \) and \( \mathbf{j} \) by

\[
\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}
\]

**EXAMPLE:**

(a) Write the vector \( \mathbf{u} = \langle 5, -8 \rangle \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

(b) If \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} \) and \( \mathbf{v} = -\mathbf{i} + 6\mathbf{j} \), write \( 2\mathbf{u} + 5\mathbf{v} \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \).
EXAMPLE:

(a) Write the vector \( \mathbf{u} = (5, -8) \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

(b) If \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} \) and \( \mathbf{v} = -\mathbf{i} + 6\mathbf{j} \), write \( 2\mathbf{u} + 5\mathbf{v} \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

Solution:

(a) \( \mathbf{u} = 5\mathbf{i} + (-8)\mathbf{j} = 5\mathbf{i} - 8\mathbf{j} \)

(b) The properties of addition and scalar multiplication of vectors show that we can manipulate vectors in the same way as algebraic expressions. Thus

\[
2\mathbf{u} + 5\mathbf{v} = 2(3\mathbf{i} + 2\mathbf{j}) + 5(-\mathbf{i} + 6\mathbf{j})
\]

\[
= (6\mathbf{i} + 4\mathbf{j}) + (-5\mathbf{i} + 30\mathbf{j})
\]

\[
= (6 - 5)\mathbf{i} + (4 + 30)\mathbf{j}
\]

\[
= \mathbf{i} + 34\mathbf{j}
\]

**Horizontal and Vertical Components of a Vector**

Let \( \mathbf{v} \) be a vector with magnitude \( |\mathbf{v}| \) and direction \( \theta \). Then \( \mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j} \), where

\[
a = |\mathbf{v}| \cos \theta \quad \text{and} \quad b = |\mathbf{v}| \sin \theta
\]

Thus, we can express \( \mathbf{v} \) as

\[
\mathbf{v} = |\mathbf{v}| \cos \theta \mathbf{i} + |\mathbf{v}| \sin \theta \mathbf{j}
\]

EXAMPLE:

(a) A vector \( \mathbf{v} \) has length 8 and direction \( \pi/3 \). Find the horizontal and vertical components, and write \( \mathbf{v} \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

(b) Find the direction of the vector \( \mathbf{u} = -\sqrt{3}\mathbf{i} + \mathbf{j} \).

Solution:

(a) We have \( \mathbf{v} = \langle a, b \rangle \), where the components are given by

\[
a = 8 \cos \frac{\pi}{3} = 8 \cdot \frac{1}{2} = 4 \quad \text{and} \quad b = 8 \sin \frac{\pi}{3} = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}
\]

Thus, \( \mathbf{v} = \langle 4, 4\sqrt{3} \rangle = 4\mathbf{i} + 4\sqrt{3}\mathbf{j} \).

(b) From the Figure on the right we see that the direction \( \theta \) has the property that

\[
\tan \theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}
\]

Thus, the reference angle for \( \theta \) is \( \pi/6 \). Since the terminal point of the vector \( \mathbf{u} \) is in Quadrant II, it follows that \( \theta = 5\pi/6 \).

EXAMPLE: A woman launches a boat from one shore of a straight river and wants to land at the point directly on the opposite shore. If the speed of the boat (relative to the water) is 10 mi/h and the river is flowing east at the rate of 5 mi/h, in what direction should she head the boat in order to arrive at the desired landing point?
EXAMPLE: A woman launches a boat from one shore of a straight river and wants to land at the point directly on the opposite shore. If the speed of the boat (relative to the water) is 10 mi/h and the river is flowing east at the rate of 5 mi/h, in what direction should she head the boat in order to arrive at the desired landing point?

Solution: We choose a coordinate system with the origin at the initial position of the boat as shown in the Figure above. Let \( \mathbf{u} \) and \( \mathbf{v} \) represent the velocities of the river and the boat, respectively. Clearly, \( \mathbf{u} = 5 \mathbf{i} \), and since the speed of the boat is 10 mi/h, we have \( |\mathbf{v}| = 10 \), so

\[
\mathbf{v} = (10 \cos \theta) \mathbf{i} + (10 \sin \theta) \mathbf{j}
\]

where the angle \( \theta \) is as shown in the Figure above. The true course of the boat is given by the vector \( \mathbf{w} = \mathbf{u} + \mathbf{v} \). We have

\[
\mathbf{w} = \mathbf{u} + \mathbf{v} = 5 \mathbf{i} + (10 \cos \theta) \mathbf{i} + (10 \sin \theta) \mathbf{j} = (5 + 10 \cos \theta) \mathbf{i} + (10 \sin \theta) \mathbf{j}
\]

Since the woman wants to land at a point directly across the river, her direction should have horizontal component 0. In other words, she should choose \( \theta \) in such a way that

\[
5 + 10 \cos \theta = 0
\]

\[
\cos \theta = -\frac{1}{2}
\]

\[
\theta = 120^\circ
\]

Thus she should head the boat in the direction \( \theta = 120^\circ \) (or N 30° W).