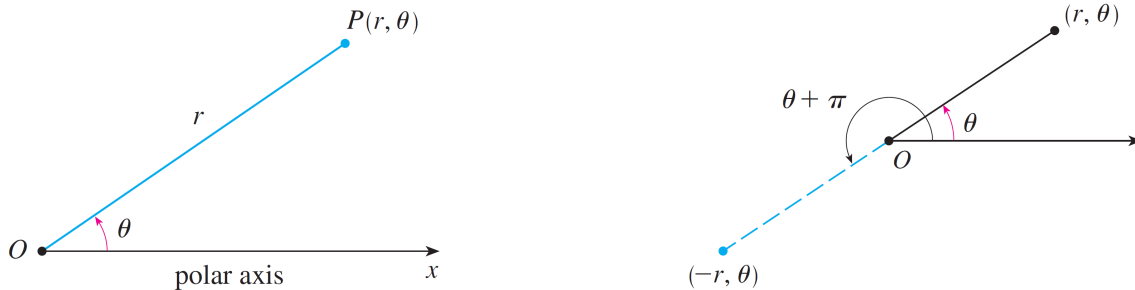


Section 8.1 Polar Coordinates

Definition of Polar Coordinates

DEFINITION: The **polar coordinate system** is a two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed point O that is called the **pole** (or origin) and an angle θ from a fixed direction. The point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates**.

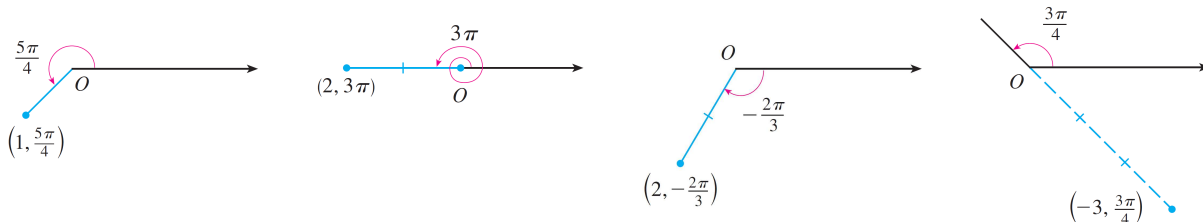
REMARK: We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that the points $(-r, \theta)$ and (r, θ) lie in the same line through O and at the same distance $|r|$ from O , but on opposite sides of O . If $r > 0$, the point (r, θ) lies in the same quadrant as θ ; if $r < 0$, it lies in the quadrant on the opposite side of the pole.



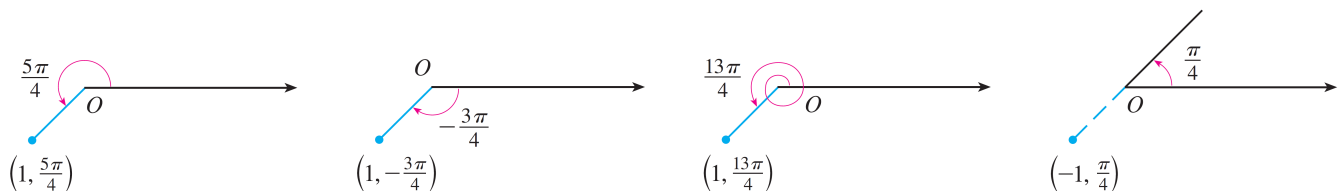
EXAMPLE: Plot the points whose polar coordinates are given:

- (a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

Solution:



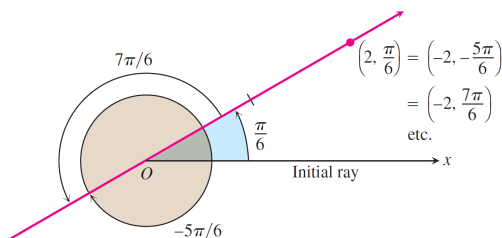
REMARK: In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ in the Example above could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$:



EXAMPLE: Find all the polar coordinates of the point $P(2, \pi/6)$.

EXAMPLE: Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution: We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$. We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.



For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \dots$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \dots$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \quad n = 0, \pm 1, \pm 2, \dots$$

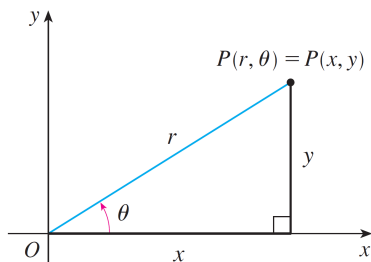
and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \quad n = 0, \pm 1, \pm 2, \dots$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on.

Relationship between Polar and Rectangular Coordinates

The connection between polar and Cartesian coordinates can be seen from the figure below and described by the following formulas:



$$x = r \cos \theta \quad y = r \sin \theta$$

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x}$$

EXAMPLE:

- Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.
- Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

EXAMPLE:

(a) Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

(b) Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Solution:

(a) We have:

$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1 \quad y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in Cartesian coordinates.

(b) If we choose r to be positive, then

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2} \quad \tan \theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $(\sqrt{2}, -\pi/4)$; another is $(\sqrt{2}, 7\pi/4)$.

Polar Equations

EXAMPLE: Express the equation $x = 1$ in polar coordinates.

Solution: We use the formula $x = r \cos \theta$.

$$x = 1$$

$$r \cos \theta = 1$$

$$r = \sec \theta$$

EXAMPLE: Express the equation $x^2 = 4y$ in polar coordinates.

Solution: We use the formulas $x = r \cos \theta$ and $y = r \sin \theta$.

$$x^2 = 4y$$

$$(r \cos \theta)^2 = 4r \sin \theta$$

$$r^2 \cos^2 \theta = 4r \sin \theta$$

$$r = 4 \frac{\sin \theta}{\cos^2 \theta} = 4 \sec \theta \tan \theta$$

EXAMPLE: Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5 \sec \theta$

(b) $r = 2 \sin \theta$

(c) $r = 2 + 2 \cos \theta$

EXAMPLE: Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5 \sec \theta$ (b) $r = 2 \sin \theta$ (c) $r = 2 + 2 \cos \theta$

Solution: We use the formulas $r^2 = x^2 + y^2$, $x = r \cos \theta$ and $y = r \sin \theta$.

(a) We have

$$r = 5 \sec \theta$$

$$r \cos \theta = 5$$

$$x = 5$$

(b) We have

$$r = 2 \sin \theta$$

$$r^2 = 2r \sin \theta$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y - 1)^2 = 1$$

(c) We have

$$r = 2 + 2 \cos \theta$$

$$r^2 = 2r + 2r \cos \theta$$

$$x^2 + y^2 = 2r + 2x$$

$$x^2 + y^2 - 2x = 2r$$

$$(x^2 + y^2 - 2x)^2 = 4r^2$$

$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$$

