Section 7.4 Trigonometric Equations

EXAMPLE: Solve the equation $\cos x = -\frac{\sqrt{2}}{2}$, and list eight specific solutions.

Solution:

**Find solutions in one period.** Because cosine has period $2\pi$, we first find the solutions in any interval of length $2\pi$. The solutions are

$$x = \frac{3\pi}{4} \quad x = \frac{5\pi}{4}$$

**Find all solutions.** Because the cosine function repeats its value every $2\pi$ units, we get all solutions of the equation by adding integer multiples of $2\pi$ to these solutions:

$$x = \frac{3\pi}{4} + 2k\pi \quad x = \frac{5\pi}{4} + 2k\pi$$

where $k$ is any integer. For $k = -1, 0, 1, 2$ we get the following specific solutions:

$$x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}, \frac{21\pi}{4}$$

EXAMPLE: Solve the equation $2\sin x - 1 = 0$. 

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Solution:

Find solutions in one period. We have

\[2 \sin x - 1 = 0\]

\[2 \sin x = 1\]

\[\sin x = \frac{1}{2}\]

Because sine has period $2\pi$, we first find the solutions in any interval of length $2\pi$. The solutions are

\[x = \frac{\pi}{6}\]
\[x = \frac{5\pi}{6}\]

Find all solutions. Because the sine function repeats its value every $2\pi$ units, we get all solutions of the equation by adding integer multiples of $2\pi$ to these solutions:

\[x = \frac{\pi}{6} + 2k\pi\]
\[x = \frac{5\pi}{6} + 2k\pi\]

where $k$ is any integer.

EXAMPLE: Solve the equation $\tan x = 2$.

Solution:

Find solutions in one period. We have

\[\tan x = 2\]

\[x = \tan^{-1} 2 \approx 1.12\]

By definition of $\tan^{-1}$, the solution that we obtained is the only solution in the interval $(-\pi/2, \pi/2)$.

Find all solutions. Because the tangent has period $\pi$, we get all solutions of the equation by adding integer multiples of $\pi$:

\[x \approx 1.12 + k\pi\]

where $k$ is any integer.

EXAMPLE: Find all solutions of the equation $\tan^2 x - 3 = 0$. 

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EXAMPLE: Find all solutions of the equation $\tan^2 x - 3 = 0$.

Solution: We have

\[
\tan^2 x - 3 = 0 \\
\tan^2 x = 3 \\
\tan x = \pm \sqrt{3}
\]

Because tangent has period $\pi$, we first find the solutions in any interval of length $\pi$. In the interval $(-\pi/2, \pi/2)$ the solutions are $x = \pi/3$ and $x = -\pi/3$. To get all solutions, we add integer multiples of $\pi$ to these solutions:

\[
x = \frac{\pi}{3} + k\pi \\
x = -\frac{\pi}{3} + k\pi
\]

where $k$ is any integer.

EXAMPLE: Solve the equation $2 \cos^2 x - 7 \cos x + 3 = 0$.

Solution: We have

\[
2 \cos^2 x - 7 \cos x + 3 = 0 \\
(2 \cos x - 1)(\cos x - 3) = 0 \\
2 \cos x - 1 = 0 \quad \text{or} \quad \cos x - 3 = 0 \\
\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = 3
\]

Because cosine has period $2\pi$, we first find the solutions in the interval $[0, 2\pi)$. For the first equation the solutions are $x = \pi/3$ and $x = 5\pi/3$. The second equation has no solution because $\cos x$ is never greater than 1. Thus the solutions are

\[
x = \frac{\pi}{3} + 2k\pi \\
x = \frac{5\pi}{3} + 2k\pi
\]

where $k$ is any integer.

EXAMPLE: Solve the equation $5 \sin x \cos x + 4 \cos x = 0$. 


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Solution: We have

$$5 \sin x \cos x + 4 \cos x = 0$$

$$\cos x(5 \sin x + 4) = 0$$

$$\cos x = 0 \quad \text{or} \quad 5 \sin x + 4 = 0$$

$$\cos x = 0 \quad \text{or} \quad \sin x = \frac{-4}{5}$$

Because sine and cosine have period $2\pi$, we first find the solutions in the interval of length $2\pi$. For the first equation the solutions in the interval $[0, 2\pi)$ are $x = \pi/2$ and $x = 3\pi/2$. The solution of the second equation is

$$x = \sin^{-1}(-0.8) \approx -0.93$$

So the solutions in an interval of length $2\pi$ are $x \approx -0.93$ and $x \approx \pi + 0.93 \approx 4.07$. We get all the solutions of the equation by adding integer multipliers to these solutions.

$$x = \frac{\pi}{2} + 2k\pi \quad x = \frac{3\pi}{2} + 2k\pi \quad x \approx -0.93 + 2k\pi \quad x \approx 4.07 + 2k\pi$$

where $k$ is any integer.